TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 227, 1977

THE TOPOLOGICAL STRUCTURE OF 4-MANIFOLDS WITH EFFECTIVE TORUS ACTIONS. I

BY

PETER SIE PAO

ABSTRACT. Torus actions on orientable 4-manifolds have been studied by F. Raymond and P. Orlik [8] and [9]. The equivariant classification problem has been completely answered there. The problem then arose as to what can be said about the topological classification of these manifolds. Specifically, when are two manifolds homeomorphic if they are not equivariantly homeomorphic? In some cases this problem was answered in the above mentioned papers. For example, if the only nontrivial stability groups are finite cyclic, then the manifolds are essentially classified by their fundamental groups. In the presence of fixed points, a connected sum decomposition in terms of S^4, $S^2 \times S^2$, CP^2, $-CP^{-2}$, $S^1 \times S^3$, and three families of elementary 4-manifolds, $R(m,n)$, $T(m,n;m',n')$, $L(n;p,q;m)$ has been obtained (where m, n, m', n', p, and q are integers). In addition, a stable homeomorphic relation for the manifolds $R(m,n)$ and $T(m,n;m',n')$ can also be found in [9]. But the topological classification of R's, T's, and especially L's were still unsolved problems. Furthermore, the connected sum decomposition of a manifold with fixed points, even in the simply connected case, is not unique.

In this paper, we completely classify the manifolds with fixed points. For the manifolds R's and T's, the above mentioned stable homeomorphic relation is proved to be the topological classification. The manifolds $L(n;p,q;m)$ form a very interesting family of 4-manifolds. They behave similarly to lens spaces. For example, the fundamental group of $L(n;p,q;m)$ is finite cyclic of order n. And it is proved that $\pi_1(L(n;p,q;m))$ and $\pi_1(L(n;p',q';m'))$ act identically on the second cohomology of their common universal covering space, $(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)$ ($n-1$ copies), even though $L(n;p,q;m)$ and $L(n;p',q';m')$ are not homotopically equivalent for some $(n;p',q';m')$'s. This family of manifolds is explicitly constructed and completely classified. In addition, a normal form is imposed on a connected sum decomposition mentioned above. These nromal forms are unique.

Most of the material of this paper appeared first in the author's doctoral dissertation. The author would like to thank his thesis advisor Professor F. Raymond for his help and encouragement.

I. Introduction.

In this introductory chapter, we shall present background material. This is done in outline form. Most of the results here are from [8] and [9].

Received by the editors October 30, 1975.
AMS (MOS) subject classifications (1970). Primary 57E15, 57A15.

Let us establish our terminology and notational conventions. By a manifold, unless otherwise specified, we mean a four dimensional closed orientable manifold. The Lie group $T^2 = SO(2) \times SO(2)$ is parametrized by (θ, ϕ) where $0 \leqslant \theta < 1, 0 \leqslant \phi < 1$. For integers m and n define $G(m,n) = \{(\theta,\phi): m\theta + n\phi = 0\}$. Note that $G(m,n)$ is isomorphic to $SO(2)$ provided that m and n are relatively prime, ($\gcd(m,n) = 1$). A manifold M is a T^2-manifold if M has at least one effective T^2-action. We shall denote the orbit space of the T^2-manifold with respect to the considered T^2-action by M^*. In a T^2-manifold, an orbit with trivial stability group will be called a principal orbit. The totality of principal orbits is denoted by P. An orbit with nontrivial finite stability group will be called an E-orbit. The totality of E-orbits is denoted by E. An orbit whose stability group is isomorphic to the circle group will be called a C-orbit and the totality of C-orbits is denoted by C. The fixed point set is denoted by F.

(I.1) *Orbit space and orbit structure*. Suppose M is a T^2-manifold. Then the following are true:

(i) M^* is a 2-manifold with boundary, where $\mathrm{Int}(M^*) = P^* \cup E^*$ and $\mathrm{Bd}(M^*) = C^* \cup F$. Furthermore, the fixed points and exceptional orbits are isolated points in M^*.

(ii) The connected component of $\mathrm{Bd}(M^*)$ is either a whole circle of C-orbits of the same orbit type, or a circle with fixed points. In the second case the points which lie on the same arc between two fixed points have the same orbit type. And if the stability groups of two consecutive arcs are $G(m_1, n_1)$ and $G(m_2, n_2)$, then $m_1 n_2 - m_2 n_1 = \pm 1$.

(I.2) *Weighted orbit space as* $C \cup F \neq \varnothing$. An orbit space together with a chosen orientation, the associated orbit structure and the slice representation for each E-orbit is called a weighted orbit space. The weighted orbit space is determined by the following orbit invariants:

$g \geqslant 0$; the genus of the oriented 2-manifold M^*.

$s \geqslant 0$; the number of boundary components of M^* all of whose orbits are C-orbits.

$t \geqslant 0$; the number of boundary components of M^* each having fixed points.

e; a specific orientation assigned to the orientable 2-manifold M^*. An orientation for T^2 is chosen once and for all, so the orientation e determines an orientation for M.

(p_i, q_i); the ith boundary component of M^*, consisting entirely of C-orbits with stability group $G(p_i, q_i)$ or equivalently $G(-p_i, -q_i)$.

$\{m,n\}_i = \{(m_{i,1}, n_{i,1}), \ldots, (m_{i,t_i}, n_{i,t_i})\}$; the ith boundary component of M^*, which contains t_i fixed points, $t_i \geqslant 2$. Here $(m_{i,j}, n_{i,j}) = \pm(m_{i,j}, n_{i,j})$ is a parametrization of the stability group of the jth oriented arc of the ith boundary component. The order of the entries in $\{m, n\}_i$ is determined up to cyclic permutation.

$(\alpha_i; \gamma_{i,1}, \gamma_{i,2}; \nu_i)$; the ith E-orbit whose stability group is finite cyclic of order α_i and generated by $(\gamma_{i,1}/\alpha_i, \gamma_{i,2}/\alpha_i)$. Here ν_i is the slice representation of the E-orbit with respect to this generator, i.e. the slice representation of the stability group Z_{α_i} is induced by the map: $(\gamma_{i,1}/\alpha_i, \gamma_{i,2}/\alpha_i) \mapsto \exp(2\pi i \nu_i/\alpha_i)$.

REMARK. For the T^2-manifolds when $F \cup C = \varnothing$, in addition to the above orbit invariants, a pair of integers (b, b') must be added in the weighted orbit space. For details see [9, I.3].

(I.3) *Equivariant classification theorem* (*P. Orlik and F. Raymond*). Let T^2 act effectively on the manifolds M and M'. Then there exists an equivariant homeomorphism h of M onto M' if and only if there exists a weight preserving homeomorphism h^* of M^* onto M'^*.

REMARK. (i) We have used the properties of smooth actions on smooth manifolds, e.g. slice representations. We can do so because all topological actions are topologically equivalent to smooth ones and smooth classification is equivalent to topological classification in this context. (See Appendix of [8].)

(ii) The above equivariant classification theorem says that the T^2-manifolds are equivariantly determined by their weighted orbit spaces. In fact, given any "legally" weighted orientable 2-manifold, M^*, there is a T^2-manifold M whose orbit space is exactly M^*. Hence there is a one-to-one correspondence between the T^2-manifolds with the specified T^2-action and the "weighted" orientable 2-manifolds. We will sometimes represent a T^2-manifold, regardless of the T^2-action, by one of its weighted orbit spaces.

(I.4) *Topological classification problem.* Since one T^2-manifold may carry more than one T^2-action, sometimes two distinct orbit spaces actually represent the same T^2-manifold. By topological classification we mean the identification and classification of the T^2-manifolds in terms of the weighted orbit spaces. This study is divided into three cases:

(i) $F \cup C = \varnothing$,

(ii) $F \neq \varnothing$, and

(iii) $F = \varnothing$, $C \neq \varnothing$.

In the case that $F \cup C = \varnothing$, the manifolds are Seifert manifolds, this problem was answered in II of [9]. Briefly speaking, "almost in all cases", two manifolds are homeomorphic if and only if they are equivariantly homeomorphic. This result was also obtained independently by H. Zieschang [11] using different methods. For case (ii), this problem was partially answered in [8] and [9]. We shall outline these partial results in this section and give a complete solution in §VI. The study of case (iii) will appear in another paper. At the present time we have only partial results.

Suppose M is a T^2-manifold and Θ: $T^2 \times M \to M$ is an effective T^2-action. Then the fixed point set, F, contains only finitely many points, and the number of the fixed points $= \chi(F) = \chi(M)$, where $\chi(F)$ and $\chi(M)$ are the Euler characteristics of F and M. (See [1, Chapter III, Theorem 10.9].) Therefore if the T^2-action Θ has fixed points then so does any other T^2-action carried by M. So the manifolds arising in the case $F \neq \varnothing$ form a distinguishable family of T^2-manifolds, (i.e. $\{F \neq \varnothing\} \cap \{F \cup C = \varnothing\} = \varnothing$ and $\{F \neq \varnothing\} \cap \{F = \varnothing, C \neq \varnothing\} = \varnothing$). We will call them T^2-manifolds with fixed points.

(I.5) *Simply connected T^2-manifolds.* Suppose M is a simply connected T^2-manifold. Then for any effective T^2-action on M, the following are true:

(i) $E = \varnothing$,

(ii) $F \neq \varnothing$, and

(iii) the orbit space M^* is homeomorphic to D^2.

Let $f_1, f_2, \ldots, f_t$ denote the fixed points, $t \geqslant 2$, and f_i^* their image in M^*. The arc, S_i^*, between f_i^* and f_{i+1}^* on $\mathrm{Bd}(M^*)$ represents an invariant 2-sphere S_i. Denote its stability group by $G(m_i, n_i)$.

Recall that

$$\begin{vmatrix} m_{i-1} & m_i \\ n_{i-1} & n_i \end{vmatrix} = e_i = \pm 1, \qquad i = 2, 3, \ldots, t,$$

$$\begin{vmatrix} m_t & m_1 \\ n_t & n_1 \end{vmatrix} = e_1 = \pm 1.$$

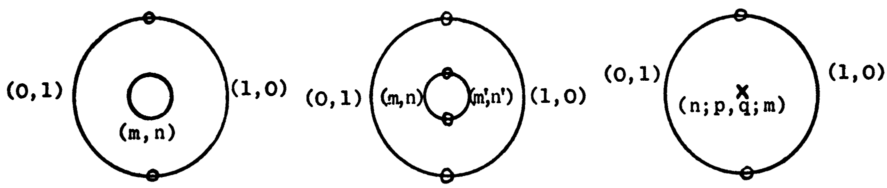
When $t = 2, 3$, and 4 we have the following table: (When $t = 4$, let $r_2 = m_1 n_3 - m_3 n_1$ and $r_3 = m_2 n_4 - m_4 n_2$.)

t	M	Condition
2	S^4	
3	CP^2	$e_1 e_2 e_3 = -1$
	$-CP^2$	$e_1 e_2 e_3 = 1$
4	$CP^2 \# CP^2$	$e_1 e_4 = -e_2 e_3 \Big\} = r_2 = 2\delta,\ e_2 e_3 \delta = 1$ or $r_2 = \delta,\ e_2 e_3 \delta = 1$
	$-CP^2 \# -CP^2$	$e_1 e_4 = -e_2 e_3 \Big\} = r_2 = 2\delta,\ e_2 e_3 \delta = -1$ or $r_2 = \delta,\ e_2 e_3 \delta = -1$
	$S^2 \times S^2$	$e_1 e_4 = e_2 e_3$, both r_2 and r_3 are even (at least one is 0)
	$CP^2 \# -CP^2$	$e_1 e_4 = e_2 e_3$, either r_2 or r_3 is odd (the other one is 0)

The above manifolds are the building blocks of the 1-connected T^2-manifolds.

THEOREM (P. ORLIK AND F. RAYMOND). *If M is a simply connected T^2-manifold, then M must be a connected sum of copies of S^4, $S^2 \times S^2$, CP^2 and $-CP^2$.*

(I.6) *T^2-manifolds with fixed points.* Let $R(m,n)$ where $\gcd(m,n) = 1$; let $T(m,n;m',n')$ where $\gcd(m,n) = \gcd(m',n') = mn' - m'n = 1$, and let $L(n;p,q;m)$ where $\gcd(n,p,q) = 1$ be the T^2-manifolds corresponding, respectively, to the following orbit spaces:



R(m,n)* T(m,n;m',n')* L(n;p,q;m)*

They form three families elementary T^2-manifolds. For the T^2-manifolds with fixed points, we have the following connected sum decomposition theorem.

THEOREM (P. ORLIK AND F. RAYMOND). *If M is a T^2-manifold with fixed points, then M must be a connected sum of copies of S^4, $S^2 \times S^2$, CP^2, $-CP^2$, $S^1 \times S^3$, and some elementary T^2-manifolds of type R, T and L.*

The following problems remained unsolved:

(i) What exactly are these elementary manifolds? Or equivalently, what are the mutual homeomorphic relations between these elementary manifolds?

(ii) The connected sum decomposition is not unique. What can be said about it?

(Both problems will be answered in this paper.)

II. Some basic notions.

(II.1) *The linear automorphisms of T^3.* Parametrize T^3 by (x,y,z) where $0 \leqslant x, y, z < 1$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be an integral matrix. Define $f(x,y,z) = (x',y',z')$ iff

$$(x',y',z') \equiv (x,y,z)A^t \quad \text{(modulo 1)}.$$

Then f is an endomorphism of T^3. In fact, if we consider R^3 as the universal covering space of T^3, then f is exactly the endomorphism of T^3 covered by the linear map represented by the matrix A. The map f is called a linear endomorphism of T^3 with associated matrix A. Let g be another linear endomorphism of T^3 with associated matrix B. It is clear that $f \circ g$ is again a linear endomorphism with associated matrix AB. Consequently, f is an automorphism of T^3 provided the matrix A is integrally invertible, or equivalently, $\det A = \pm 1$.

Let Θ and Θ' be two T^3-actions on T^3, such that

Θ: $(\theta, \phi, \psi) \times (x,y,z) \to (x + \theta, y + \phi, z + \psi)$, and

Θ': $(\theta, \phi, \psi) \times (x,y,z) \to (x + a_{11}\theta + a_{12}\phi + a_{13}\psi, y + a_{21}\theta + a_{22}\phi + a_{23}\psi, z + a_{31}\theta + a_{32}\phi + a_{33}\psi)$, where $\det(a_{ij}) = \pm 1$. Let f be the linear automorphism of T^3 with the associated matrix (a_{ij}). Then f is an equivariant homeomorphism with respect to the T^3-actions Θ and Θ'. Now consider two T^2-actions, $\underline{\Theta}$ and $\underline{\Theta}'$, on T^3, such that

$c\underline{\Theta}$: $(\theta, \phi) \times (x,y,z) \to (x + \theta, y + \phi, z)$, and

$\underline{\Theta}'$: $(\theta, \phi) \times (x,y,z) \to (x + a_{11}\theta + a_{12}\phi, y + a_{21}\theta + a_{22}\phi, z + a_{31}\theta + a_{32}\phi)$.

Since the T^2-actions $\underline{\Theta}$ and $\underline{\Theta}'$ are obtained by restricting, respectively, the T^3-actions Θ and Θ' to a T^2 subgroup, the linear automorphism f above is also T^2-equivariant with respect to $\underline{\Theta}$ and $\underline{\Theta}'$. In fact, we may construct new $(\underline{\Theta} - \underline{\Theta}')$ equivariant automorphisms simply by changing the third column of the matrix (a_{ij}) and requiring that the determinant of the new matrix be ± 1.

(II.2) *The linear self-homeomorphisms of* $D^2 \times T^2$. Represent the element of $D^2 \times T^2$ by $(\rho, x; y, z)$, where $0 \leqslant \rho \leqslant 1$, $0 \leqslant x, y, z < 1$, and $(0, x; y, z) = (0, 0; y, z)$ for all x, y, z. Notice that $\mathrm{Bd}(D^2 \times T^2) = \{(1, x; y, z)\colon 0 \leqslant x, y, z < 1\}$ is homeomorphic to T^3, and the parametrization adopted here is exactly the one introduced in (II.1). Let f be a linear endomorphism of $\mathrm{Bd}(D^2 \times T^2) = T^3$ with associated matrix A, where A is an integral matrix of the following form:

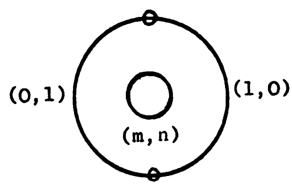
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Then f can be extended continuously to $D^2 \times T^2$ by defining $f(\rho, x; y, z) = (\rho, x'; y', z')$, where $0 \leqslant x', y', z' < 1$ and $(x', y', z') \equiv (x, y, z)A^t \pmod 1$. f is called a linear map on $D^2 \times T^2$ with associated matrix A. Similarly, if g is another linear map of $D^2 \times T^2$ with associated matrix B, then $f \circ g$ is linear with associated matrix AB. And f is a self-homeomorphism of $D^2 \times T^2$ provided $\det A = \pm 1$.

(II.3) *Topological attaching.* Let (X, A), (Y, B) be two topological pairs and $f, g\colon A \to B$ be two homeomorphisms. Then we can form two topological spaces $X \cup_f Y$ and $X \cup_g Y$. Suppose F: $(X,A) \to (X,A)$ and G: $(Y,B) \to (Y,B)$ are homeomorphisms, such that $G \circ f = g \circ F$. Then $X \cup_f Y$ is homeomorphic to $X \cup_g Y$ by $F * G$, where $F * G$ maps elements of X by F and elements of Y by G.

In the case that X, Y are differentiable manifolds, $A = \mathrm{Bd}(X)$, $B = \mathrm{Bd}(Y)$, and f, g, F, G are diffeomorphisms, the map $F * G$: $X \cup_f Y \to X \cup_g Y$ is a diffeomorphism.

III. The manifolds of type R. Let $R(m,n)$, m, n relatively prime, be the T^2-manifold with the weighted orbit space



Recall from 4.2 and 4.5 of [9] that:

THEOREM III.1 (P. ORLIK AND F. RAYMOND).

$$R(m,1) = \begin{cases} S^2 \times S^2 \# S^3 \times S^1 & \text{if } m \text{ is even,} \\ CP^2 \# -CP^2 \# S^3 \times S^1 & \text{if } m \text{ is odd;} \end{cases}$$

and

THEOREM III.2 (P. ORLIK AND F. RAYMOND). *Given relatively prime integers m and n, there exists a nonnegative integer k, such that*

$$R(m,n) \# k(S^2 \times S^2) = \begin{cases} S^2 \times S^2 \# S^3 \times S^1 \# k(S^2 \times S^2) & \text{if } mn \equiv 0 \pmod 2, \\ CP^2 \# -CP^2 \# S^3 \times S^1 \# k(S^2 \times S^2) & \text{if } mn \equiv 1 \pmod 2. \end{cases}$$

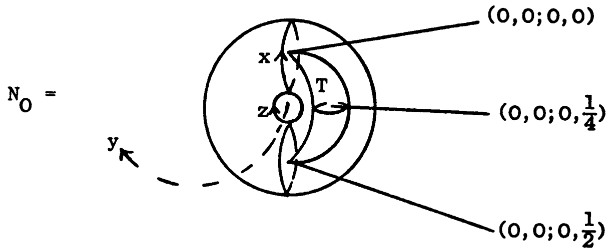
In this section, our main theorem is the following.

THEOREM III.3.

$$R(m,n) = \begin{cases} S^2 \times S^2 \# S^3 \times S^1 & \text{if } mn \equiv 0 \pmod 2, \\ CP^2 \# -CP^2 \# S^3 \times S^1 & \text{if } mn \equiv 1 \pmod 2. \end{cases}$$

Clearly $S^2 \times S^2 \# S^3 \times S^1$ and $CP^2 \# -CP^2 \# S^3 \times S^1$ do not have the same homotopy type. It follows that Theorem III.3 completely classifies the manifolds of type R.

Let N_0, N_1 be $D^2 \times T^2$ and parametrized as in (II.2). Let T be the subset of N_0, $\{(\rho, x; y, z)$: $z \in [0, \frac{1}{2}]$; $\rho = 2tz$ if $z \in [0, \frac{1}{4}]$ and $\rho = 2t(1/2 - z)$ if $z \in [\frac{1}{4}, \frac{1}{2}]$, where $t \in [0,1]\}$. T is homeomorphic to $S^1 \times D^3$. We can consider N_0 as $(D^2 \times S^1) \times S^1$, where $D^2 \times S^1$ is represented by the entries ρ, x, z, and S^1 is represented by the entry y. N_0 and T can be pictured as follows:



For relatively prime integers m, n and integers a, b such that $an - bm = -1$, let f: $\mathrm{Bd}(N_0) \to \mathrm{Bd}(N_1)$ be the linear homeomorphism, (cf. II.1) with the associated matrix

$$\begin{pmatrix} a & b & 0 \\ m & n & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can write the point in $D^2 \times S^2$ in the form $(\delta, y; \rho, x, z)$, where $0 \leqslant \delta \leqslant 1$, $0 \leqslant x, y, z < 1$, and

$$\rho = \begin{cases} 2z & \text{if } z \in [0, \frac{1}{4}], \\ 2(\frac{1}{2} - z) & \text{if } z \in [\frac{1}{4}, \frac{1}{2}], \end{cases}$$

satisfying the relation: $(0,0;0,0,z) = (0,y;0,x,z)$ for all x, y and z. The above rather peculiar parametrization of $D^2 \times S^2$ should be interpreted as follows: (δ, y) is a point in D^2 given in terms of polar coordinate, and (ρ, x, z) is the above mentioned parametrization of the points on the boundary of the ball in T. Briefly: $D^2 \times S^2$ is considered as the "complement" of $T \subset N_0$ in the sense of "surgery". Let h: $\mathrm{Bd}(D^2 \times S^2) \to \mathrm{Bd}(T)$ be the homeomorphism defined by $h(1, y; \rho, x, z) = (\rho, x; y, z)$. For the above defined h and f we have the following:

LEMMA III.4. $R(m,n) = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_f N_1$.

PROOF. Let $X = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_f N_1$. Clearly X is an orientable manifold. By the definition of $R(m,n)$, it is sufficient to show that X supports an effective T^2-action which gives the desired weighted orbit space.

Let T^2 act on N_0 by $(\theta, \phi) \times (\rho, x; y, z) \to (\rho, x + \theta; y + \phi, z)$, T^2 act on N_1 by $(\theta, \phi) \times (\rho, x; y, z) \to (\rho, x + a\theta + b\phi; y + m\theta + n\phi, z)$, and T^2 act on D^2

$\times S^2$ by $(\theta, \phi) \times (\delta, y; \rho, x, z) \to (\delta, y + \phi; \rho, x + \theta, z)$. We have the following:

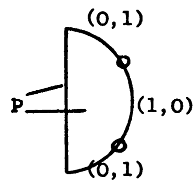
(i) The T^2-actions on N_0, N_1 and $D^2 \times S^2$ defined above are all effective.

(ii) $T \subset N_0$ is invariant under the above T^2-action. Hence $N_0 - \mathrm{Int}(T)$ is also invariant. We shall consider $N_0 - \mathrm{Int}(T)$ as a T^2-manifold with boundary.

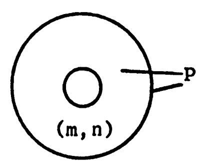
(iii) Both h and f defined above are T^2-equivariant.

Therefore X supports an effective T^2-action and the orbit space, $X^* = (D^2 \times S^2)^* \cup_{h*} (N_0 - \mathrm{Int}(T))^* \cup_{f*} N_1^*$. It is easily seen that:

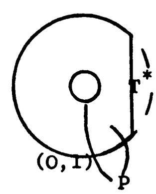
$(D^2 \times S^2)^* =$



$N_1^* =$

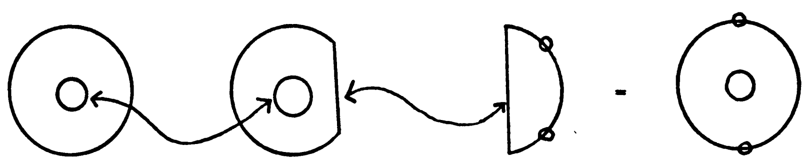


and $N_0^* =$



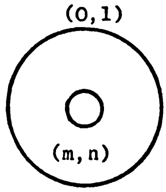
So

$X^* =$



The lemma is proved.

Let N_0, N_1 and f be as above and $Q(m, n) = N_0 \cup_f N_1$. Then $Q(m, n)$ is a T^2-manifold, and with respect to the T^2-actions defined above, the orbit space $Q(m, n)^*$ is



Consider the subset $T \subset N_0$ defined above, it is in fact a T^2-invariant tubular neighborhood of the orbit passing through the point $(0,0;0,\frac{1}{4})$. By Lemma III.4 above, to obtain the manifold $R(m,n)$, we first remove $\mathrm{Int}(T)$, (which is homeomorphic to $\mathrm{Int}(S^1 \times D^3)$), from $Q(m,n)$ and then attach a copy of $D^2 \times S^2$ to $(Q(m,n) - \mathrm{Int}(T))$ along their boundaries by the identity map. Moreover, the T^2-action on $D^2 \times S^2$ was defined so that the above attaching map is equivariant. Specifically: The T^2-action on $\mathrm{Bd}(T) = S^1 \times S^2$ can be considered as the product action of the standard S^1-action on S^1 and the standard S^1-action on S^2. (By the standard S^1-action on S^2, we mean the induced S^1-action on S^2 when we consider S^1 as a subgroup of $SO(3)$ in the usual manner.) And the aforementioned T^2-action on $D^2 \times S^2$ can be considered as the product action of the standard S^1-action on D^2 and the standard S^1-action on S^2. The above implies that the T^2-manifold $R(m,n)$ is obtained from the T^2-manifold $Q(m,n)$ by an equivariant surgery of type (2, 3). The manifold $Q(m,n)$ will be studied in a later paper. It is proved there that $Q(m,n)$ is homeomorphic to $S^1 \times L(m,n)$, where $L(m,n)$ is a three-dimensional lens space. As a conclusion we have the following corollary.

Corollary III.5. *The T^2-manifold $R(m,n)$ can be obtained from $S^1 \times L(m,n)$ by a single T^2-equivariant surgery of type* (2, 3).

The following is immediate:

Lemma III.6. *For any given relatively prime integers m and n, $R(m,n) = R(n,m)$.*

Lemma III.7. *Given relatively prime integers m and n, then $R(m,n) = R(m,n')$ if $n \equiv n' \pmod m$ and $(n-n')/m \equiv 0 \pmod 2$, or $n \equiv -n' \pmod m$ and $(n+n')/m \equiv 0 \pmod 2$.*

Proof. By Lemma III.4 we may assume that $R(m,n) = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_f N_1$, and $R(m,n') = D^2 \times S^2 \cup_{h'} (N_0 - \mathrm{Int}(T)) \cup_{f'} N_1$.

Suppose $n \equiv n' \pmod m$. Let $n - n' = mk$ and a, b be some integers such that $an - bm = -1$. Then $an' - (b - ak)m = a(n' + mk) - bm = an - bm = -1$. The aforementioned maps f and f' can be chosen to be the linear homeomorphisms with the associated matrices

$$\begin{pmatrix} a & b & 0 \\ m & n & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b-ak & 0 \\ m & n' & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. Let $F'\colon N_0 \to N_0$ be the linear homeomorphism with the associated matrix

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that $T \subset N_0$ is invariant under F'. So $F'|_{(N_0-T)}$ is a self-homeomorphism of $N_0 - T$, which we will also denote by F'. Assume that F' can be extended as a homeomorphism $F\colon D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \to D^2 \times S^2 \cup_{h'} (N_0 - \mathrm{Int}(T))$. Since $F|_{\mathrm{Bd}((D^2\times S^2)\cup_h(N_0-\mathrm{Int}(T)))} = F'|_{\mathrm{Bd}(N_0)}$ and the associated matrix of $f' \circ F|_{\mathrm{Bd}((D^2\times S^2)\cup_h(N_0-\mathrm{Int}(T)))} = f' \circ F'|_{\mathrm{Bd}(N_0)}$ is

$$\begin{pmatrix} a & b-ak & 0 \\ m & n' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ m & n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the associated matrix of the map $\mathrm{id}_{N_1} \circ f$, it follows that $f' \circ F|_{\mathrm{Bd}((D^2\times S^2)\cup_h(N_0-\mathrm{Int}(T)))} = \mathrm{id}_{N_1} \circ f$. Consequently, $R(m,n)$ is homeomorphic to $R(m,n')$ by the homeomorphism $F * \mathrm{id}_{N_1}$ (cf. II.3). We shall now show that F' can be extended as a homeomorphism $F\colon D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \to D^2 \times S^2 \cup_{h'} (N_0 - \mathrm{Int}(T))$, when $k = n - n'/m$ is even. Consider the map $h'^{-1} \circ F' \circ h\colon \mathrm{Bd}(D^2 \times S^2) \to \mathrm{Bd}(D^2 \times S^2)$, which is a self-homeomorphism of $S^1 \times S^2$. Clearly, it is sufficient to show that when k is even, the map $h'^{-1} \circ F' \circ h$ can be extended homeomorphically to $D^2 \times S^2$. It is easy to see that $h'^{-1} \circ F' \circ h$ can be extended homeomorphically to $D^2 \times S^2$ provided that it is isotopic to the identity map. Observe that $h^{-1} \circ F' \circ h(1,y;\rho,x,z) = (1,y;\rho,x+ky,z)$. Recall the parametrization of $\mathrm{Bd}(D^2 \times S^2) = S^1 \times S^2$: A point on $\mathrm{Bd}(D^2 \times S^2) = S^1 \times S^2$ is represented by $(1,y;\rho,x,z)$, where $(1,2\pi y)$ are the ordinary polar coordinates of a point on the unit circle S^1, and $(\rho\cos 2\pi x, \rho\sin 2\pi x, z)$ are the ordinary coordinates of a point in S^2. So in terms of these coordinates, the map $h'^{-1} \circ F' \circ h$ can be considered as:

$$\begin{aligned} &h'^{-1} \circ F' \circ h(\cos 2\pi y, \sin 2\pi y; \rho\cos 2\pi x, \rho\sin 2\pi x, z) \\ &\qquad = (\cos 2\pi y, \sin 2\pi y; \rho\cos 2\pi(x+ky), \rho\sin 2\pi(x+ky), z). \end{aligned}$$

Let $S\colon S^1 \to SO(3)$ be the map defined by

$$S(\cos 2\pi y, \sin 2\pi y) = \begin{pmatrix} \cos 2\pi(ky) & -\sin 2\pi(ky) & 0 \\ \sin 2\pi(ky) & \cos 2\pi(ky) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $0 \leqslant y < 1$. Then

$$\begin{aligned} &h'^{-1} \circ F' \circ h(\cos 2\pi y, \sin 2\pi y; \rho \cos 2\pi x, \rho \sin 2\pi x, z) \\ &\quad = (\cos 2\pi y, \sin 2\pi y; (\rho \cos 2\pi x, \rho \sin 2\pi x, z) \cdot (S(\cos 2\pi y, \sin 2\pi y))^t), \\ &\qquad\qquad \text{for } (\rho \cos 2\pi x, \rho \sin 2\pi x, z) \cdot (S(\cos 2\pi y, \sin 2\pi y))^t \\ &\quad = (\rho \cos 2\pi x, \rho \sin 2\pi x, z) \begin{pmatrix} \cos 2\pi ky & \sin 2\pi ky & 0 \\ -\sin 2\pi ky & \cos 2\pi ky & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad = (\rho(\cos 2\pi x \cdot \cos 2\pi ky - \sin 2\pi x \cdot \sin 2\pi ky), \\ &\qquad\qquad \rho(\cos 2\pi x \cdot \sin 2\pi ky + \sin 2\pi x \cdot \cos 2\pi ky), z) \\ &\quad = (\rho \cos 2\pi(x + ky), \rho \sin 2\pi(x + ky), z). \end{aligned}$$

We say that the map $h'^{-1} \circ F' \circ h$ is induced by the map S.

DEFINITION. A map f: $S^1 \times S^n \to S^1 \times S^n$ is induced by a map S: $S^1 \to SO(n + 1)$, if $f(x,y) = (x, y \cdot S(x)^t)$, where $x \in S^1$ and $y \in S^n$. The following proposition is then immediate:

PROPOSITION. *Suppose the homeomorphism f: $S^1 \times S^n \to S^1 \times S^n$ is induced by a map S: $S^1 \to SO(n + 1)$, then f is isotopic to the identity map provided S is homotopically trivial.*

So we need to show that the map S: $S^1 \to SO(3)$ defined above is homotopically trivial. Consider the imbedding i: $S^1 = SO(2) \to SO(3)$, such that

$$i(A) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right), \quad \text{where } A \in SO(2).$$

It is well known that $i_\# : \pi_1(SO(2)) \to \pi_1(SO(3))$ is surjective. In fact $i_\#(g) = 0$, where $g \in \pi_1(SO(2))$, iff g has even index, i.e. $\pi_1(SO(2))/\langle g \rangle$ is a finite cyclic group of even order. Notice that the map l: $S^1 \to SO(2)$ such that

$$l(\cos x, \sin x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$$

is a representative of a generator $[l]$ in $\pi_1(SO(2))$, and the map l^k is a representative of the element $[l^k] \in \pi_1(SO(2))$, where

$$l^k(\cos x, \sin x) = \begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix}.$$

Since $S = i \circ l^k$, and $[S] = [i \circ l^k] = i_\#[l^k]$, it follows that $[S]$ is homotopically trivial provided k is even. We conclude that $R(m,n)$ is homeomorphic to $R(m,n')$, if $n \equiv n' \pmod m$ and $(n - n')/m$ is even. A similar argument yields that $R(m,n) = R(m,n')$ when $n \equiv -n' \pmod m$ and $(n + n')/m$ is even.

Finally, we are ready to prove *Theorem* II.3. The proof is divided into three steps. We shall first prove the following sublemma.

SUBLEMMA. Given relatively prime integers m and n, $m \neq \pm 1$, then there exists a positive integer s, $s < |m|$, such that $R(m,n) = R(m,s)$.

PROOF. Let $n \equiv s \pmod m$ and $n - s = mk$, where $0 < s < |m|$. Suppose k is even, then by *Lemma* III.7, $R(m,n) = R(m,s)$. We are done. Suppose k is odd. Notice that $n + m - s = m(k+1)$ and $n - m - s = m(k-1)$. Since both $k + 1$ and $k - 1$ are even, *Lemma* III.7 implies that

$$R(m,n) = R(m,m-s) \quad \text{and} \quad R(m,n) = R(m,-s-m).$$

If $m > 0$, then $0 < m - s < m = |m|$. If $m < 0$, then $0 < -s - m < |m|$. The lemma is proved.

Secondly, we shall show that for any given relatively prime integers m and n, there exists an integer t, such that $R(m,n) = R(t,1)$. Then it follows from *Theorem* III.1 that $R(m,n)$ is homeomorphic to either $S^2 \times S^2 \# S^1 \times S^3$ or $CP^2 \# -CP^2 \# S^1 \times S^3$.

In the case that $m = \pm 1$, since $R(\pm 1, n) = R(\pm n, 1)$, the above assertion is valid. Suppose $m \neq \pm 1$. By the sublemma above we may assume $0 < n < |m|$. If $n = 1$ we are done, otherwise applying *Lemma* III.6 and the above sublemma, we can find an integer m' such that $0 < m' < n$ and $R(m,n) = R(n,m) = R(n,m')$. If $m' = 1$ we are done, otherwise we can again apply *Lemma* III.6 and the sublemma to find an integer n' such that $0 < n' < m'$ and $R(m,n) = R(n,m') = R(m',n')$. We can repeat this reduction until one of the integer entries equals 1. The assertion is proved.

Finally, suppose $mn \equiv 0 \pmod 2$ and $R(m,n) = CP^2 \# -CP^2 \# S^1 \times S^3$. Then the quadratic form of $R(m,n)$ is of type I. Consequently, the quadratic form of $R(m,n) \# k(S^2 \times S^2)$ is also of type I, for any positive integer k. However, this contradicts *Theorem* III.2, which says that

$$R(m,n) \# k(S^2 \times S^2) = S^2 \times S^2 \# S^1 \times S^3 \# k(S^2 \times S^2)$$

for some integer k, where the quadratic form of

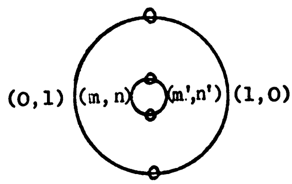
$$S^2 \times S^2 \# S^1 \times S^3 \# k(S^2 \times S^2)$$

is of type II. Therefore $R(m,n)$ must be $S^2 \times S^2 \# S^1 \times S^3$. Similarly we can prove that $R(m,n) = CP^2 \# -CP^2 \# S^1 \times S^3$ if mn is odd. This completes the proof of Theorem III.3.

IV. **The manifolds of type** T. Let $T(m,n;m',n')$, where

$$\gcd(m,n) = \gcd(m',n') = 1 \quad \text{and} \quad mn' - m'n = \pm 1,$$

be the T^2-manifold with the following weighted orbited space:



Since $mn' - m'n = \pm 1$, at least two of these four integers must be odd and one is even. Define $t = 0$ if there are two even integers, and $t = 1$ otherwise. Recall from 4.6 of [9] that:

THEOREM IV.1 (P. ORLIK AND F. RAYMOND). *If one of the integers m, n, m' and n' equals ± 1, then*

$$T(m,n;m',n') = \begin{cases} S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3 & \text{if } t = 0, \\ CP^2 \# -CP^2 \# CP^2 \# -CP^2 \# S^1 \times S^3 & \text{if } t = 1, \end{cases}$$

and

THEOREM IV.2 (P. ORLIK AND F. RAYMOND). *For any given integers m, n, m' and n' such that* $\gcd(m,n) = 1$, $\gcd(m',n') = 1$ *and* $mn' - m'n = \pm 1$, *there exists an integer k such that*

$$\begin{aligned} &T(m,n;m',n') \# k(S^2 \times S^2) \\ &\quad = \begin{cases} S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3 \# k(S^2 \times S^2) & \text{if } t = 0, \\ CP^2 \# -CP^2 \# CP^2 \# -CP^2 \# S^1 \times S^3 \# k(S^2 \times S^2) & \\ & \text{if } t = 1. \end{cases} \end{aligned}$$

In this section our main result is the following:

THEOREM IV.3.

$$T(m,n;m',n') = \begin{cases} S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3 & \text{if } t = 0, \\ CP^2 \# -CP^2 \# CP^2 \# -CP^2 \# S^1 \times S^3 & \text{if } t = 1. \end{cases}$$

Since $S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3$ and $CP^2 \# -CP^2 \# CP^2 \# -CP^2 \# S^1 \times S^3$ do not have the same homotopy type, this theorem completely classifies the manifolds of type T.

The proof of this theorem is similar to the argument used in proving *Theorem* III.3. We shall refer the reader to §III for the details.

Let N_0, N_1, $T_0 \subset N_0$, f, $(D^2 \times S^2)_0$, and h_0 be defined as in §III, where T_0, $(D^2 \times S^2)_0$ and h_0 take the place of T, $D^2 \times S^2$ and h. Define $T_1 \subset N_1$, $(D^2 \times S^2)_1$, and h_1: $\mathrm{Bd}(D^2 \times S^2)_1 \to \mathrm{Bd}(T_1)$ in the same way as it was for $T_0 \subset N_0$, $(D^2 \times S^2)_0$ and h_0: $\mathrm{Bd}(D^2 \times S^2)_0 \to \mathrm{Bd}(T_0)$. Let f: $\mathrm{Bd}(N_0) \to \mathrm{Bd}(N_1)$ be the linear homeomorphism with the associated matrix

$$\begin{pmatrix} m' & n' & 0 \\ m & n & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An argument similar to the one used in proving Lemma III.4 yields the following:

LEMMA IV.4.

$$T(m,n;m',n') = [(D^2 \times S^2)_0 \cup_{h_0} (N_0 - \mathrm{Int}(T_0))] \cup_f [(D^2 \times S^2)_1 \cup_{h_1} (N_1 - \mathrm{Int}(T_1))].$$

Notice that

$$(D^2 \times S^2)_0 \cup_{h_0} (N_0 - \mathrm{Int}(T_0)) \cup_f N_1$$

is homeomorphic to $R(m,n)$. (Cf. *Lemma* III.4.) This lemma implies that $T(m,n;m',n')$ is the manifold obtained by doing an equivariant surgery of type (2, 3) on the T^2-manifold $R(m,n)$.

COROLLARY IV.5. *The T^2-manifold $T(m,n;m',n')$ is obtained from $S^2 \times S^2 \# S^1 \times S^3$ or $CP^2 \# -CP^2 \# S^1 \times S^3$ by doing a single equivariant surgery of type* (2,3).

The following is immediate.

LEMMA IV.6. $T(m,n;m',n') = T(n',m';n,m)$.

LEMMA IV.7. *Given integers m, n, m', n' such that* $\gcd(m,n) = 1$, $\gcd(m',n') = 1$, $mn' - m'n = \pm 1$, *then $T(m,n;m',n')$ is homeomorphic to $T(m,s;m',s')$ if $(n-s)/m = (n'-s')/m' = 2k$, or $(n+s)/m = (n'+s')/m' = 2k$, for some integer k.*

PROOF. By *Lemma* IV.4 we may assume that

$$T(m,n;m',n') = [(D^2 \times S^2)_0 \cup_{h_0} (N_0 - \mathrm{Int}(T_0))]$$
$$\cup_f [(D^2 \times S^2)_1 \cup_{h_1} (N_1 - \mathrm{Int}(T_1))],$$

and

$$T(m,s;m',s') = [(D^2 \times S^2)_0 \cup_{h_0} (N_0 - \mathrm{Int}(T_0))]$$
$$\cup_{f'} [(D^2 \times S^2)_1 \cup_{h_1} (N_1 - \mathrm{Int}(T_1))],$$

where h_0, h_1 are the attaching maps defined previously, and f, f': $\mathrm{Bd}(N_0) \to \mathrm{Bd}(N_1)$, are the linear homeomorphisms with the following associated matrices:

$$\begin{pmatrix} m' & n' & 0 \\ m & n & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m' & s' & 0 \\ m & s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose $n - s = 2km$ and $n' - s' = 2km'$. Let F': $(N_0 - \mathrm{Int}(T_0)) \to (N_0 - \mathrm{Int}(T_0))$ be the linear homeomorphism with the associated matrix

$$\begin{pmatrix} 1 & 2k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then it follows from the proof of case I of *Lemma* III.7 that the map F' can be extended as a homeomorphism

$$F\colon (D^2 \times S^2)_0 \cup_{h_0} (N_0 - \mathrm{Int}(T_0)) \to (D^2 \times S^2)_0 \cup_{h_0} (N_0 - \mathrm{Int}(T_0)).$$

Moreover, since

$$\begin{pmatrix} 7' & s' & 0 \\ m & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} m' & 2km' + s' & 0 \\ m & 2km + s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} m' & n' & 0 \\ m & n & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it follows that

$$f' \circ F|_{\mathrm{Bd}((D^2 \times S^2)_0 \cup_{h_0} (N_0 - \mathrm{Int}(T_0)))} = \mathrm{id}_{(D^2 \times S^2)_1 \cup_{h_1} (N_1 - \mathrm{Int}(T_1))} \circ f.$$

Hence $T(m,n;m',n')$ is homeomorphic to $T(m,s;m',s')$ by the homeomorphism $F * \mathrm{id}$ as asserted. Similarly one can show that $T(m,n;m',n')$ is homeomorphic to $T(m,s;m',s')$ if $n + s/m = n' + s'/m \equiv O(2)$.

PROOF OF THEOREM IV.3. The proof is divided into three steps. Firstly, by *Lemma* IV.7 and the argument used in proving the sublemma of §III, one can prove the following:

SUBLEMMA. (1) If $m \neq \pm 1$, then there exists a positive integer s, $s < |m|$, such that $T(m,n;m',n') = T(m,s;m',s')$ for some integer s'.

(2) If $m' \neq \pm 1$, then there exists a positive integer s', $s' < |m|$, such that $T(m,n;m',n') = T(m,s;m',s')$ for some integer s.

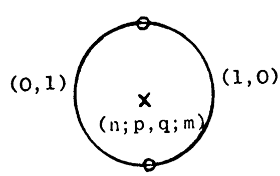
Secondly, we want to establish a reduction on the integer entries m, n, m', n', such that by repeating this reduction we can make one of these four integer entries equal 1. Then it follows from *Theorem* IV.1 that $T(m,n;m',n')$ is homeomorphic to either $S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3$ or $CP^2 \# -CP^2 \# CP^2 \# -CP^2 \# S^1 \times S^3$. In the case $m = \pm 1$, we are done. Suppose $m \neq \pm 1$, then by (1) of the above sublemma, we may assume $0 < n < |m|$. If $n = 1$ then we are done, otherwise applying *Lemma* IV.6 and (2) of the above sublemma, we can find integers s and s', $0 < s < |n|$, such that $T(m,n;m',n') = T(n',m';n,m) = T(n',s';n,s)$. If $s = 1$ then we are done. Otherwise again applying *Lemma* IV.6 and (1) of the sublemma, we can find integers t and t', $0 < t < s$, such that $T(n',s';n,s) = T(s,n;s',n') = T(s,t;s',t')$. Since $t < s < n < |m|$, by repeating this reduction, we can make one of these integers entries equal 1.

Finally, suppose $t = 0$, where t is the number defined in IV.1, and

$$T(m,n;m',n') = CP^2 \# -CP^2 \# CP^2 \# -CP^2 \# S^1 \times S^3.$$

Then the quadratic form of $T(m,n;m',n') \# k(S^2 \times S^2)$ is of type I, for any nonnegative integer k. This contradicts *Theorem* IV.2, which says that $T(m,n;m',n') \# k(S^2 \times S^2)$ is homeomorphic to $S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3 \# k(S^2 \times S^2)$ for some integer k, where the quadratic form of $S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3 \# k(S^2 \times S^2)$ is of type II. Therefore $T(m,n;m',n')$ must be $S^2 \times S^2 \# S^2 \times S^2 \# S^1 \times S^3$. Similarly we can prove that when $t = 1$, $T(m,n;m',n') = CP^2 \# -CP^2 \# CP^2 \# -CP^2 \# S^1 \times S^3$. This completes the proof of Theorem IV.3.

V. **The manifolds of type** L. In this section we study $L(n;p,q;m)$, the T^2-manifold corresponding to the weighted orbit space



where m, n, p, q are some integers such that $\gcd(m,n) = \gcd(n,p,q) = 1$. Unlike the T^2-manifolds of types R and T, the manifolds of type L form a family of manifolds which is not known to us. (Since their fundamental groups are finite cyclic, they can not be the connected sum or product of any well-known manifolds.) In this section we shall construct these manifolds explicitly and classify them completely. Our main results are the following:

THEOREM V.1. *Given integers n, p, q, m as above, the manifold $L(n;p,q;m)$ is homeomorphic to either $L(n;0,1;1)$ or $L(n;1,1;1)$.*

Depending upon whether n is even or odd, we have the following:

THEOREM V.2. *If n is odd, the manifolds $L(n;0,1;1)$ and $L(n;1,1;1)$ are homeomorphic. If n is an even integer, then the manifolds $L(n;0,1;1)$ and $L(n;1,1;1)$ do not have the same homotopy type.*

Since the group $\pi_1(L(n;p,q;m))$ is finite cyclic of order n, it follows from the above two theorems that the family of the L-manifolds contains one manifold for each odd integer greater than 2, and two distinct manifolds for each even integer greater than 1. The proof of these two theorems is based on the following ten lemmas, (*Lemma* V3–*Lemma* V.12).

The following are immediate:

LEMMA V.3. *Given integers n, p, q, m as above, let t be an integer such that* $\gcd(n,t) = 1$, *then $L(n;p,q;m) = L(n;tp,tq;tm)$ and*

LEMMA V.4. $L(n;p,q;m) = L(n;q,p;m)$.

LEMMA V.5. *Given a finite cyclic subgroup H of T^2, if H is of order n, then there exist relatively prime integers p and q such that $(p/n,q/n)$ generates H.*

PROOF. Choose a generator $(s/n,t/n)$ of H. (This is possible since any element in T^2 of order n can be written in this form.) Then clearly $\gcd(n,s,t) = 1$. In fact, if $\gcd(n,s,t) = k$, then the order of $(s/n,t/n)$ would be n/k. Let a, b be some integers such that $a \cdot \gcd(s,t) + bn = 1$. So we have $a \cdot \gcd(s,t) \equiv 1 \pmod{n}$. Let $p = s/\gcd(s,t)$ and $q = t/\gcd(s,t)$. Observe that in T^2, the element $a(s/n,t/n) = (as/n,at/n) = (a \cdot \gcd(s,t)p/n, a \cdot \gcd(s,t)q/n)$ $= a \cdot \gcd(s,t)(p/n,q/n) = (p/n,q/n)$. The integers p and q are obviously relatively prime. Since $\gcd(a,n) = 1$, it follows that $(p/n,q/n)$ is a generator of H. The lemma is proved.

It follows immediately from this lemma that any L-manifold $L(n;s,t;r)$ is T^2-equivalent to some $L(n;p,q;m)$, where p and q are relatively prime. From now on, unless otherwise specified, for a manifold of type L, we shall assume that its second and third integer entries are relatively prime.

A simple computation gives us the following lemma:

LEMMA V.6. *Given integers p, q, a_0, b_0 such that $a_0p + b_0q = 1$, then in T^2 the subgroup $G(na_0, nb_0) \cap G(na_0 + q, nb_0 - p)$ is finite cyclic of order n and generated by the element $(p/n, q/n)$.*

Let N_0, N_1, $T \subset N_0$ and h: $\mathrm{Bd}(D^2 \times S^2) \to \mathrm{Bd}(T)$ be defined as in III, and let f: $\mathrm{Bd}(N_0) \to \mathrm{Bd}(N_1)$ be the linear homeomorphism with the associated matrix A,

$$A = \begin{pmatrix} ma_0 & mb_0 & \alpha \\ na_0 & nb_0 & \beta \\ na_0 + q & nb_0 - p & 0 \end{pmatrix},$$

where $a_0p + b_0q = 1$ and $\alpha n - \beta m = 1$. (Notice that $\det(A) = -1$, so the map f is indeed a homeomorphism.) The following lemma should be treated as an explicit construction of the manifold $L(n; p, q; m)$.

LEMMA V.7. $L(n; p, q; m) = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_f N_1$.

PROOF. Let $X = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_f N_1$. As in *Lemma* III.4, it is sufficient to construct a T^2-action on X with the desired weighted orbit space. Define a T^2-action on X as follows:

on $D^2 \times S^2$: $(\theta, \phi) \times (\delta, y; \rho, x, z) \to (\delta, y + \phi; \rho, x + \theta, z)$;

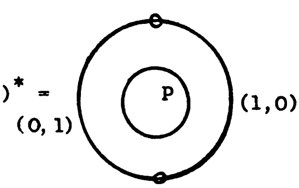
on $N_0 - \mathrm{Int}(T)$: $(\theta, \phi) \times (\rho, x; y, z) \to (\rho, x + \theta; y + \phi, z)$;

on N_1: $(\theta, \phi) \times (\rho, x; y, z) \to (\rho, x + ma_0\theta + mb_0\phi; y + na_0\theta + nb_0\phi,$

$$z + (na_0 + q)\theta + (nb_0 - p)\phi).$$

(Since the attaching maps h and f are all equivariant with respect to the above defined T^2-actions, this does give a T^2-action on X.) It is seen in III.4 that

$(D^2 \times S^2 \cup_h (N_0 - \mathrm{int}(T)))^* =$



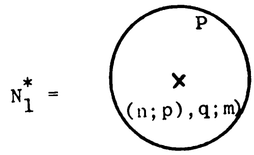
For the orbit space N_1^*, the following conclusions are immediate:

(1) N_1^* is homeomorphic to D^2.

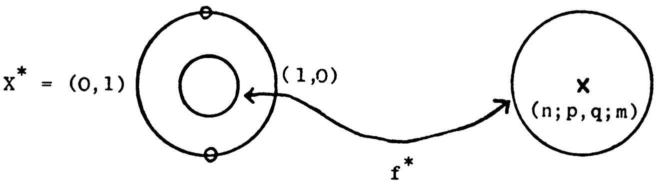
(2) All the orbits in N_1 are principle orbits, except the subset, $\{(0, x; y, z)$: $0 \leqslant x, y, z < 1\}$, which is an E-orbit.

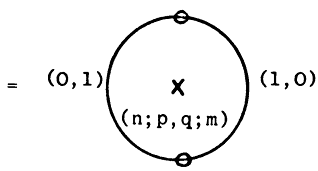
(3) From *Lemma* II.3.6 the stability group of the above E-orbit is finite cyclic of order n and generated by $(p/n, q/n)$.

(4) With respect to the generator $(p/n, q/n)$, the slice representation of this E-orbit is m.
(In fact, N_1 is a "Tube" of the above E-orbit.) In conclusion, we have



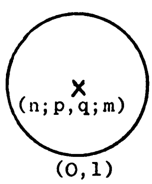
Therefore,





The lemma is proved.

Denote $S(n;p,q;m)$ by $N_0 \cup_f N_1$. Then $S(n;p,q;m)$ is the T^2-manifold corresponding to the weighted orbit space



As in *Lemma* III.5, we can see that the T^2-manifold $L(n;p,q;m)$ is obtained from the T^2-manifold $S(n;p,q;m)$ by doing a T^2-equivariant surgery of type (2, 3). It will be shown in a later paper that $S(n;p,q;m)$ is homeomorphic to the product of S^1 and some three-dimensional lens space, here S^3 and $S^1 \times S^2$ are considered as some special lens spaces. In particular, $S(n;0,1;1) = S(n;1,1;1) = S^1 \times S^3$. Combining this with Theorem V.1 we have the following corollary.

COROLLARY V.8. *The manifold of type L is the manifold resulting from a T^2-equivariant surgery of type* $(2,3)$ *on* $S^1 \times S^3$.

It is known that by doing surgery of type (2, 3) on one element in the fundamental group of the manifold, we can produce at most two distinct resulting manifolds. Thus the above corollary can be also viewed as a definition of the manifolds of type L.

LEMMA V.9. *$L(n;p,q;m) = L(n;p,q;m')$, if $m' \equiv -m \pmod n$.*

PROOF. Let $m' + m = kn$. By *Lemma* V.7, let $L(n;p,q;m) = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_f N_1$, and $L(n;p,q;m') = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_{f'} N_1$, where the attaching maps f and f' are the linear homeomorphisms with the associated matrices

$$\begin{pmatrix} ma_0 & mb_0 & \alpha \\ na_0 & nb_0 & \beta \\ na_0 + q & nb_0 - p & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m'a_0 & m'b_0 & \alpha - k\beta \\ na_0 & nb_0 & -\beta \\ na_0 + q & nb_0 - p & 0 \end{pmatrix},$$

respectively. Let $F'\colon N_0 - \mathrm{Int}(T) \to N_0 - \mathrm{Int}(T)$ and $G\colon N_1 \to N_1$ be the linear homeomorphisms with the associated matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -k & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

respectively. Notice that $F'|_{\mathrm{Bd}(T)}(\rho,x;y,z) = (\rho,-x;-y,z)$, and F' can be extended as a homeomorphism

$$F\colon D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \to D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)),$$

by defining $F(\delta,y;\rho,x,z) = (\delta,-y;\rho,-x,z)$ for all points $(\delta,y;\rho,x,z)$ in $D^2 \times S^2$. Since

$$\begin{aligned}
&\begin{pmatrix} 1 & -k & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} ma_0 & mb_0 & \alpha \\ na_0 & nb_0 & \beta \\ na_0 + q & nb_0 - p & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} ma_0 - kna_0 & mb_0 - knb_0 & \alpha - k\beta \\ -na_0 & -nb_0 & -\beta \\ -(na_0 + q) & -(nb_0 - p) & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} -m'a_0 & -m'b_0 & \alpha - k\beta \\ -na_0 & -nb_0 & -\beta \\ -(na_0 + q) & -(nb_0 - p) & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} m'a_0 & m'b_0 & \alpha - k\beta \\ na_0 & nb_0 & -\beta \\ na_0 + q & nb_0 - p & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

it follows that $G \circ f = f' \circ F$. Therefore $F * G$ is a homeomorphism from $L(n;p,q;m)$ to $L(n;p,q;m')$. The lemma is proved.

Lemma V.10. *$L(n;p,q;m) = L(n;p+2kq,q;m)$ for all integers k.*

Proof. As in *Lemma* V.7, let $L(n;p,q;m) = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_f N_1$, and $L(n;p+2kq,q;m) = D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \cup_{f'} N_1$, where f and f' are the linear homeomorphisms with the associated matrices

$$\begin{pmatrix} ma_0 & mb_0 & \alpha \\ na_0 & nb_0 & \beta \\ na_0 + q & nb_0 - p & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} ma_0 & m(b_0 - 2a_0 k) & \alpha \\ na_0 & n(b_0 - 2a_0 k) & \beta \\ na_0 + q & n(b_0 - 2a_0 k) - (p + 2kq) & 0 \end{pmatrix},$$

respectively. Define $F'\colon (N_0 - \mathrm{Int}(T)) \to (N_0 - \mathrm{Int}(T))$ to be the linear homeomorphism with the associated matrix

$$\begin{pmatrix} 1 & 2k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then as shown in the proof of *Lemma* III.4, the map F' can be extended as a homeomorphism

$$F\colon D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)) \to D^2 \times S^2 \cup_h (N_0 - \mathrm{Int}(T)).$$

Notice that

$$\begin{pmatrix} ma_0 & m(b_0 - 2a_0 k) & \alpha \\ na_0 & n(b_0 - 2a_0 k) & \beta \\ na_0 + q & n(b_0 - 2a_0 k) - (p + 2kq) & 0 \end{pmatrix} \begin{pmatrix} 1 & 2k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ma_0 & mb_0 & \alpha \\ na_0 & nb_0 & \beta \\ na_0 + q & nb_0 - p & 0 \end{pmatrix},$$

so $f' \circ F = \mathrm{id}_{N_1} \circ f$. It follows that $F * \mathrm{id}$ is a homeomorphism from $L(n;p,q;m)$ to $L(n;p+2kq,q;m)$. The lemma is proved.

Lemma V.11. *$L(n;p,q;m) = L(n;p,q+2kp;m)$ for all integers k.*

PROOF.

$$\begin{aligned} L(n;p,q;m) &= L(n;q,p;m) && \text{(Lemma V.4)} \\ &= L(n;q+2kp,p;m) && \text{(Lemma V.10)} \\ &= L(n;p,q+2kp;m) && \text{(Lemma V.4).} \end{aligned}$$

LEMMA V.12. *Given integers n, p, q, m as above, there exists an integer m' such that $L(n;p,q;m)$ is homeomorphic to either $L(n;0,1;m')$ or $L(n;1,1;m')$.*

PROOF. Let $a = \gcd(n,p)$ and $b = \gcd(n,q)$. Then a and b are relatively prime and $ab|n$. Let c equal the product of the prime factors of n/ab which divide a, and d equal the product of the prime factors of n/ab which divide b. That is, $c = n/ab \cdot k$, where k is the largest factor of n/ab such that $\gcd(k,a) = 1$, and $d = n/ab \cdot k'$, where k' is the largest factor of n/ab such that $\gcd(k',b) = 1$. Notice that c and d are relatively prime and $abcd|n$. Let $e = n/abcd$, then clearly $\gcd(e,p) = \gcd(e,q) = 1$. We need the following:

SUBLEMMA. Let n, p, q, a, b, c, d, e be the above integers. Then $\gcd(q + 2ep, n) = 1$ if p is even, and $\gcd(p + 2eq, n) = 1$ if p is odd.

Assuming this sublemma, the proof of *Lemma* V.12 proceeds as follows: Suppose p is odd. It follows from *Lemma* V.10 and the above sublemma that $L(n;p,q;m) = L(n;p+2eq,q;m)$ and $\gcd(p+2eq,n) = 1$. Let s be an integer such that $s(p+2eq) \equiv 1 \pmod n$, then in T^2 the element

$$s((p+2eq)/n, q/n) = (1/n, sq/n).$$

Since s and n are relative prime, it follows from *Lemma* V.3 that $L(n;p,q;m) = L(n;p+2eq,q;m) = L(n;1,sq;sm)$. By *Lemma* V.11 and *Lemma* V.4, we can conclude that:

$$L(n;1,sq;sm) = \begin{cases} L(n;1,0;sm) = L(n;0,1;sm) & \text{if } sq \text{ is even,} \\ L(n;1,1;sm) \quad \text{if } sq \text{ is odd .} \end{cases}$$

Suppose p is even. Then by *Lemma* V.11 and the above sublemma, we have $L(n;p,q;m) = L(n;p,q+2ep;m)$ and $\gcd(q+2ep,n) = 1$. Let t be an integer that $t(q+2ep) \equiv 1 \pmod n$, then in T^2 the element $t(p/n,(q+2ep)/n) = (tp/n, 1/n)$. Since t and n are relatively prime, *Lemma* V.3 implies $L(n;p,q;m) = L(n;p,q+2ep;m) = L(n;tp,1;tm)$. Applying *Lemma* V.10, we have

$$L(n;tp,1;tm) = \begin{cases} L(n;0,1;tm) & \text{if } tp \text{ is even,} \\ L(n;1,1;tm) & \text{if } tp \text{ is odd.} \end{cases}$$

The lemma is proved.

PROOF OF THE SUBLEMMA. Suppose p is even. p and q are relatively prime, so q must be odd. Let s be a prime number or 1 such that $s|\gcd(q + 2ep, n)$. Then it is easily seen that one of the following three assertions must be true:

(1) $s|a$ and $s|(q + 2ep)$;

(2) $s|b$ and $s|(q + 2ep)$;

(3) $s|e$ and $s|(q + 2ep)$.

Suppose (1) is true, then it follows immediately that $s|p$ and $s|q$. Consequently, s must be 1. Suppose (2) is true, then we have $s|q$ and $s|2ep$. q is odd, so s cannot be 2; it follows that $s|ep$. But the integers ep and q are relatively prime, so s is 1. In the case (3) is true, we have $s|e$ and $s|q$. But $\gcd(e,q) = 1$, so s must be 1 again. In conclusion: the only common factor of $q + 2ep$ and n is 1, that is $\gcd(q + 2ep, n) = 1$.

Suppose p is odd and $s|\gcd(p + 2eq, n)$. Similar to the previous case, one of the following three assertions must be valid:

(1) $s|(p + 2eq)$ and $s|a$;

(2) $s|(p + 2eq)$ and $s|b$;

(3) $s|(p + 2eq)$ and $s|e$.

(1) implies $s|p$ and $s|2eq$. Since p is odd s cannot be 2; it follows that $s|eq$. But $\gcd(p,eq) = 1$, so $s = 1$. (2) implies $s|p$ and $s|q$, which again implies that $s = 1$. (3) implies $s|p$ and $s|e$. But $\gcd(e,p) = 1$, so $s = 1$. As in the previous case, the above argument proved that $\gcd(p + 2eq, n) = 1$. This completes the proof of the sublemma.

We are now ready to prove one of our main results, *Theorem* V.1. By *Lemma* V.12, it is sufficient to show for any positive integer n that the manifolds $L(n;0,1;m)$ and $L(n;1,1;m)$ are homeomorphic to either $L(n;0,1;1)$ or $L(n;1,1;1)$. We shall first consider the manifold $L(n;0,1;m)$: Suppose m is even, then the following conclusions are immediate:

$$\begin{aligned} L(n;0,1;m) &= L(n;m,1;m) \qquad \text{(Lemma V.10)} \\ &= L(n;1,q';1) \text{ for some integer } q', \qquad \text{(Lemma V.3)} \\ &= \begin{cases} L(n;1,0;1) & \text{if } q' \text{ is even,} \\ L(n;1,1;1) & \text{if } q' \text{ is odd,} \end{cases} \qquad \text{(Lemma V.11).} \end{aligned}$$

Suppose m is odd. When n is also an odd number, then it follows from *Lemma* V.9 that $L(n;0,1;m) = L(n;0,1;n-m)$, where $n - m$ is an even integer. We are back to the previous case. Suppose m is odd and n is even. Then

$$\begin{aligned} L(n;0,1;m) &= L(n;2,1;m) \qquad \text{(Lemma V.10)} \\ &= L(n;2,m;m) \qquad \text{(Lemma V.11)} \\ &= L(n;p',1;1) \text{ for some integer } p', \qquad \text{(Lemma V.3)} \\ &= \begin{cases} L(n;0,1;1) & \text{if } p' \text{ is even,} \\ L(n;1,1;1) & \text{if } p' \text{ is odd,} \end{cases} \qquad \text{(Lemma V.10).} \end{aligned}$$

Next, let us consider the manifold $L(n; 1, 1; m)$. Suppose m is odd, then

$$\begin{aligned} L(n; 1, 1; m) &= L(n; m, 1; m) \qquad \text{(Lemma V.10)} \\ &= L(n; 1, q'; 1) \text{ for some integer } q', \qquad \text{(Lemma V.3)} \\ &= \begin{cases} L(n; 1, 0; 1) & \text{if } q' \text{ is even,} \\ L(n; 1, 1; 1) & \text{if } q' \text{ is odd,} \end{cases} \qquad \text{(Lemma V.11).} \end{aligned}$$

Suppose m is even. It follows from *Lemma* V.9 that

$$L(n; 1, 1; m) = L(n; 1, 1; n - m).$$

Since n and m are relatively prime, the integer n must be odd, it follows that $n - m$ is odd. We are back to the previous case. This completes the proof of *Theorem* V.1.

Since in T^2, $(0/n, 1/n)$ and $(n/n, 1/n)$ represent the same element, it follows that $(n; 0, 1; 1)$ and $(n; n, 1; 1)$ represent the same orbit invariant. Hence $L(n; 0, 1; 1) = L(n; n, 1; 1)$. In the case that n is an odd integer, by *Lemma* V.10 we have $L(n; 0, 1; 1) = L(n; n, 1; 1)$. This proves the first part of *Theorem* V.2. In the case that n is an even number, the manifolds $L(n; 0, 1; 1)$ and $L(n; 1, 1; 1)$ have different mod 2 cohomology ring structure. We shall study this in the rest of this section.

The covering spaces of the L-manifolds. Let N_0, N_1 denote two copies of $D^2 \times T^2$, parametrized as in (II.2). Define $T_i \subset N_0$, $i = 0, 1, \ldots, n - 1$, to be the set of elements $(\rho, x; y, z)$ where ρ and z satisfy the following conditions:

$$(3i + 1)/3n \leqslant z \leqslant (3i + 2)/3n, \quad \text{and for } 0 \leqslant t \leqslant 1,$$

$$(*) \qquad \rho = \begin{cases} 2t(z - (3i + 1)/3n) & \text{if } (3i + 1)/3n \leqslant z \leqslant (3i + 1)/3n + 1/6n, \\ 2t((3i + 2)/3n - z) & \text{if } (3i + 1)/3n + 1/6n \leqslant z \leqslant (3i + 2)/3n. \end{cases}$$

Let T^2 act on N_0 by $(\theta, \phi) \times (\rho, x; y, z) \to (\rho, x + \theta; y + \phi, z)$, and on N_1 by $(\theta, \phi) \times (\rho, x; y, z) \to (\rho, x; y + \theta, z + \phi)$. Then the following assertions are immediate:

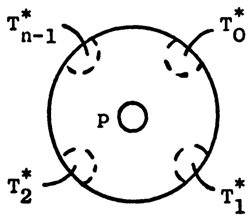
(1) The linear homeomorphism $f\colon \mathrm{Bd}(N_0) \to \mathrm{Bd}(N_1)$ with the associated matrix

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

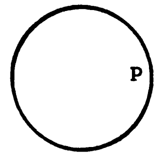
is T^2-equivariant.

(2) For each $i = 0, 1, \ldots, n - 1$, T_i is a T^2-invariant subset of N_0.

(3) The weighted orbit spaces of N_0 and N_1 are



and



respectively.

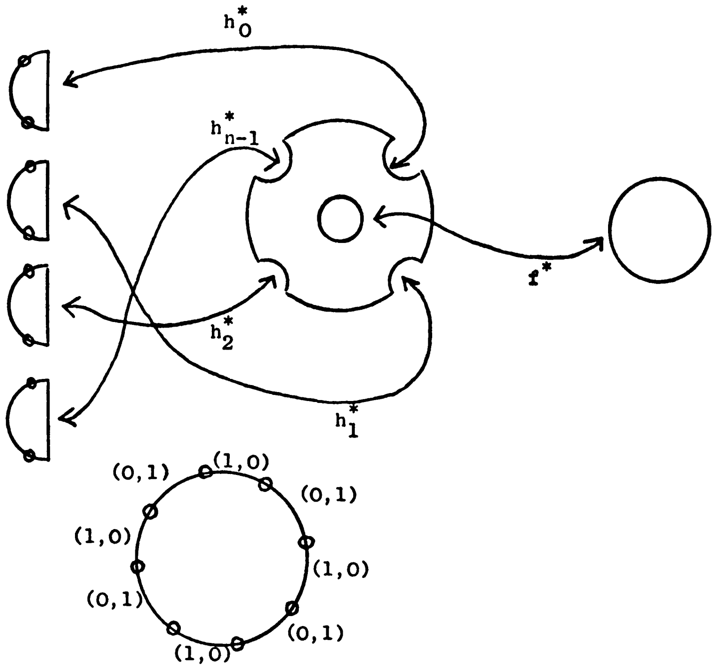
Represent an element in $(D^2 \times S^2)_i$, $i = 0, 1, \ldots, n-1$, by $(\delta, y; \rho, x, z)$, where $0 \leqslant \delta \leqslant 1$, $0 \leqslant x, y < 1$ and ρ, z satisfy the conditions $(*)$ above. Let T^2 act on $(D^2 \times S^2)_i$, $i = 0, 1, \ldots, n-1$, by $(\theta, \phi) \times (\delta, y; \rho, x, z) \to (\delta, y + \phi; \rho, x + \theta, z)$, and let h_i: $\mathrm{Bd}((D^2 \times S^2)_i) \to \mathrm{Bd}(T_i)$, $i = 0, 1, \ldots, n-1$, be the homeomorphism such that $h_i(1, y; \rho, x, z) = (\rho, x; y, z)$. Then clearly h_i is T^2-equivariant, for $i = 0, 1, \ldots, n-1$. Let

$$\tilde{L}_n = \bigcup_{i=0}^{n-1} (D^2 \times S^2)_i$$

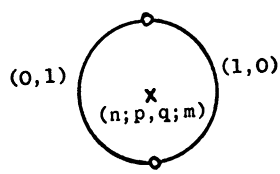
$$\cup_{\{h_i\}} \left(N_0 - \bigcup_{i=0}^{n-1} \mathrm{Int}(T_i) \right) \cup_f N_1 .$$

Then $\tilde{L}_n$ is a T^2-manifold and

$\tilde{L}^*_n =$



Checking the table of (I.5) it is easily seen that L_n is homeomorphic to $\#\,(n-1)(S^2 \times S^2)$. Now, consider the Z_n-action on $\tilde{L}_n$ induced by the homeomorphism w, where on N_1: $w(\rho, x; y, z) = (\rho, x + m/n; y + p/n, z + q/n)$; on $N_0 - \bigcup_{i=0}^{n-1} \operatorname{Int}(T_i)$: $w(\rho, x; y, z) = (\rho, x + p/n; y + q/n; z - m/n)$; and on $(D^2 \times S^2)_i$, $(i = 0, 1, \ldots, n-1)$: $w(\delta, y; \rho, x, z) = (\delta, y + q/n; \rho, x + p/n, z - mn)$. ($w$ commutes with the attaching maps f and h_i, $(i = 0, 1, \ldots, n-1)$, and w is of order n, so w indeed induces a Z_n-action on $\tilde{L}_n$.) Notice that this Z_n-action commutes with the T^2-action on $\tilde{L}_n$, and the Z_n action is in fact an orientation preserving, free action. Therefore the Z_n-orbit space $\tilde{L}_n/Z_n$ is a manifold with a naturally induced T^2-action. Since $(\tilde{L}_n/Z_n)/T^2$ is homeomorphic to $(\tilde{L}_n/T^2)/Z_n$, a simple computation shows that the T^2-weighted orbit space of $\tilde{L}_n/Z_n$ is



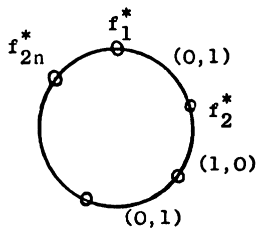
So $\tilde{L}_n/Z_n = L(n; p, q; m)$. As a conclusion we have the following:

THEOREM V.13. *Given integers n, p, q, m such that* $\gcd(m, n) = 1$ *and* $\gcd(n, p, q) = 1$, *the universal covering space of the manifold* $L(n; p, q; m)$ *is* $\#\,(n-1)(S^2 \times S^2)$.

REMARK. (1) A by-product of the above argument is that $\pi_1(L(n; p, q; m)) = Z_n$.

(2) If we can assume that $\pi_1(L(n; p, q; m)) = Z_n$, we can prove the above theorem very easily: Since the T^2-action on $L(n; p, q; m)$ has fixed points, it follows from the Covering Action Theorem that there exists a T^2-action on $\tilde{L}_n$ such that the covering map is T^2-equivariant. (See [1, Chapter I, Theorem 9.1], or [2, Theorem 3.1].) $\tilde{L}_n$ is an n-fold covering of $L(n; p, q; m)$, so $\tilde{L}_n$ has $2n$ fixed points. Let p: $\tilde{L}_n \to \tilde{L}_n^*$ be the orbit map of the covering T^2-action. Since $P_\#: \pi_1(\tilde{L}_n) \to \pi_1(\tilde{L}_n^*)$ is surjective, $\tilde{L}_n^*$ is simply connected. Consequently,

$\tilde{L}_n^*$ =



Hence $\tilde{L}_n = \#\,(n-1)(S^2 \times S^2)$.

The direct geometric proof of *Theorem* V.13 has the advantage of giving us a clear description of the deck transformation of $\pi_1(L(n;p,q;m))$ on $\tilde{L}_n$. A further study of this gives the following interesting corollary.

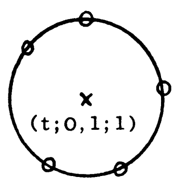
COROLLARY V.14. *Let n, s and t be positive integers such that $n = st$. Let $\bar{L}_n$ and $\bar{L}'_n$ be the covering spaces of $L(n;0,1;1)$ and $L(n;1,1;1)$ corresponding to the subgroups of $\pi_1(L(n;0,1;1))$ and $\pi_1(L(n;1,1;1))$ of order t respectively. Then*

$$\bar{L}_n = (\#\ (s-1)(S^2 \times S^2))\ \#\ L(t;0,1;1),$$

and

$$\bar{L}'_n = (\#\ (s-1)(S^2 \times S^2))\ \#\ L(t;1,1;1).$$

PROOF. Notice that $\bar{L}_n = \tilde{L}_n/\langle w^s \rangle$, where w is the generator of the group Z_n defined previously and $\langle w^s \rangle$ is the subgroup generated by w^s. By an easy computation, we can see that the orbit space of $\bar{L}_n$ with respect to the naturally induced T^2-action is



, (2s fixed points).

Again checking the table of (I.5), we have $\bar{L}'_n = (\#\ (s-1)(S^2 \times S^2))\ \#\ L(t;0,1;1)$. Using the same argument, we can show that

$$\bar{L}_n = (\#\ (s-1)(S^2 \times S^2))\ \#\ L(t;1,1;1).$$

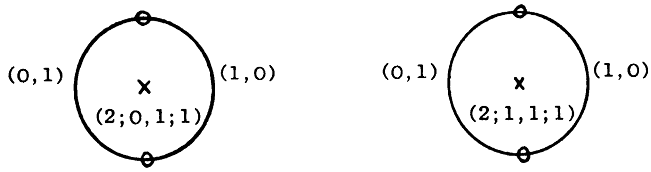
Parametrize $S^2 \times S^2$ by $((x,y,z),(x',y',z'))$, where $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 = 1$. Let ϕ and ψ be the self-homeomorphisms of $S^2 \times S^2$ such that

$$\phi((x,y,z),(x',y',z')) = ((x,y,-z),(-x',-y',-z')),$$

and

$$\psi((x,y,z),(x',y',z')) = ((-x,-y,-z),(-x',-y',-z')).$$

Since ϕ and ψ are both orientation preserving homeomorphisms of order two, it follows that the quotient spaces $(S^2 \times S^2)/\phi$ and $(S^2 \times S^2)/\psi$ are both orientable 4-manifolds with fundamental groups Z_2. Define a T^2-action on $S^2 \times S^2$ by taking the product of two standard circle actions on S^2. It is easily seen that on $S^2 \times S^2$ this T^2-action commutes with the Z_2 actions induced by the homeomorphisms ϕ and ψ. Therefore the 4-manifolds $(S^2 \times S^2)/\phi$ and $(S^2 \times S^2)/\psi$ have T^2-actions, and by a simple computation, it follows that the orbit spaces of $(S^2 \times S^2)/\phi$ and $(S^2 \times S^2)/\psi$ are



respectively. We have the following corollary.

COROLLARY V.15.

$$L(2;0,1;1) = (S^2 \times S^2)/\phi, \quad \textit{and} \quad L(2;1,1;1) = (S^2 \times S^2)/\psi.$$

REMARK. Observe that the manifolds $(S^2 \times S^2)/\phi$ and $(S^2 \times S^2)/\psi$ are S^2-bundles over RP^2. If we let ε and η denote the trivial and the nontrivial line bundles over RP^2 respectively, it is easily seen that $(S^2 \times S^2)/\phi$ and $(S^2 \times S^2)/\psi$ are in fact the sphere bundles of the 3-plane bundles $\varepsilon \oplus \varepsilon \oplus \eta$ and $\eta \oplus \eta \oplus \eta$ respectively, where "$\oplus$" stands for the Whitney sum of bundles, (cf. [7, p. 2]).

The homology and cohomology of the L-manifolds. From now on, as a matter of convenience, for each integer greater than 1 we shall let L_n and L'_n denote the T^2-manifolds $L(n;0,1;1)$ and $L(n;1,1;1)$ respectively. The integral homology and cohomology of L_n and L'_n are the same. We shall study them for the manifold L_n; the same argument works for L'_n. Since L_n is orientable, by the Poincaré Duality and Hurewicz Isomorphism Theorem, we have

$$H_0(L_n) = H^4(L_n) = Z, \qquad H^0(L_n) = H_4(L_n) = Z,$$

and

$$H^3(L_n) = H_1(L_n) = \pi_1(L_n) = Z_n.$$

It follows from the Universal Coefficient Theorem that $\mathrm{Free}(H^1(L_n)) = \mathrm{Free}(H_1(L_n)) = 0$, and $\mathrm{Tor}(H^1(L_n)) = \mathrm{Tor}(H_0(L_n)) = 0$. (Given a module G, $\mathrm{Free}(G)$ means the free part of G, and $\mathrm{Tor}(G)$ means the torsion part of G.) Hence $H^1(L_n) = H_3(L_n) = 0$. Since the T^2-action on L_n has two fixed points, it follows from [1, Chapter III, Theorem 10.3] that $\chi(L_n) = 2$. However, $\chi(L_n) = 2 + rk(H^2(L_n))$, so $rk(H^2(L_n)) = 0$, that is $\mathrm{Free}(H^2(L_n)) = 0$. Hence

$$H_2(L_n) = H^2(L_n) = \mathrm{Tor}(H^2(L_n)) = \mathrm{Tor}(H_1(L_n)) = Z_n.$$

We have the following:

THEOREM V.16. *Given an integer n, $n \geqslant 2$, the integral homology and cohomology of the manifolds L_n and L'_n are the same. They are described in the table below*:

Z	0	1	2	3	4
H^*	Z	0	Z_n	Z_n	Z
H_*	Z	Z_n	Z_n	0	Z

REMARK. (1) The cohomology ring structures of L_n and L'_n are obviously trivial.

(2) Applying the Universal Coefficient Theorem, we can find Z_m-cohomology and Z_m-homology of L_n and L'_n. They are shown in the table below:

Z_m	0	1	2	3	4
H^*	Z_m	$Z_{\gcd(n,m)}$	$Z_{\gcd(n,m)} \oplus Z_{\gcd(n,m)}$	$Z_{\gcd(n,m)}$	Z_m
H_*	Z_m	$Z_{\gcd(n,m)}$	$Z_{\gcd(n,m)} \oplus Z_{\gcd(n,m)}$	$Z_{\gcd(n,m)}$	Z_m

In contrast to the integral cohomology, the Z_m-cohomology ring structure of L_n and L'_n are not trivial. We shall next study the Z_2-cohomology ring of L_{2k} and L'_{2k}.

Recall from *Corollary* V.15 that the manifolds L_2 and L'_2 are the sphere bundles $\varepsilon \oplus \varepsilon \oplus \eta$ and $\eta \oplus \eta \oplus \eta$ over RP^2 respectively. Let E denote the disk bundle of $\varepsilon \oplus \varepsilon \oplus \eta$ and E' denote the disk bundle of $\eta \oplus \eta \oplus \eta$. We have the bundle pairs $(E, L_2) \xrightarrow{p} RP^2$ and $(E', L'_2) \xrightarrow{p'} RP^2$. Let $\beta \in H^3(E, L_2; Z_2)$, $\beta' \in H^3(E', L'_2; Z_2)$ be the Thom classes, and

$$\alpha \in H^1(E; Z_2) = H^1(RP^2; Z_2) = Z_2,$$
$$\alpha' \in H'(E'; Z_2) = H^1(RP^2; Z_2) = Z_2$$

be the generators. Consider the Z_2-cohomology long exact sequence of the pair (E, L_2):

$$\cdots \to H^r(E, L_2) \xrightarrow{j_r} H^r(E) \xrightarrow{i_r} H^r(L_2) \xrightarrow{\delta_r} H^{r+1}(E, L_2) \to \cdots.$$

Since

$$H^r(E, L_2) = \begin{cases} Z_2 & \text{if } r = 3, 4, 5, \\ 0 & \text{otherwise,} \end{cases} \quad \text{(Thom Isomorphism Theorem),}$$

and

$$H^r(E) = H^r(RP^2) = \begin{cases} Z_2 & \text{if } r = 0, 1, 2, \\ 0 & \text{otherwise,} \end{cases}$$

we have the following short exact sequences:

$$0 \to H^1(E) \xrightarrow{i_1} H^1(L_2) \to 0,$$

$$0 \to H^2(E) \xrightarrow{i_2} H^2(L_2) \xrightarrow{\delta_2} H^3(E, L_2) \to 0$$

and

$$0 \to H^3(L_2) \xrightarrow{\delta_3} H^4(E, L_2) \to 0,$$

$$0 \to H^4(L_2) \xrightarrow{\delta_4} H^5(E, L_2) \to 0.$$

Notice that: $H^1(E; Z_2)$ is generated by α,
$H^2(E; Z_2)$ is generated by α^2,
$H^3(E, L_2; Z_2)$ is generated by β,
$H^4(E, L_2; Z_2)$ is generated by $\alpha\beta$,
$H^5(E, L_2; Z_2)$ is generated by $\alpha^2\beta$, and $\alpha^3 = 0$.
Let $u = i_1(\alpha) \in H^1(L_2; Z_2)$ and $v \in H^2(L_2; Z_2)$ such that $\delta_2(v) = \beta$. (There are two such elements in $H^2(L_2; Z_2)$; choose v to be either one of them.) Then by the exactness of the above short exact sequences, we have: $u^2 \neq 0$, $u^3 = 0$, u^2, v generates $H^2(L_2; Z_2)$, uv generates $H^3(L_2; Z_2)$, and u^2v generates $H^4(L_2; Z_2)$. The same argument works for the manifold L_2': Define $u' \in H^1(L_2'; Z_2)$ and $v' \in H^2(L_2'; Z_2)$ as in the L_2 case. Then $u'^2 \neq 0$, $u'^3 = 0$, u'^2, v' generates $H^2(L_2'; Z_2)$, $u'v'$ generates $H^3(L_2'; Z_2)$, and u'^2v' generates $H^4(L_2'; Z_2)$.

To determine $H^*(L_2; Z_2)$ and $H^*(L_2'; Z_2)$ we need to know v^2 and v'^2. Since both δ_4 and δ_4' are isomorphisms, $v^2 = \mathrm{Sq}^2 v = 0$ iff $\delta_4(\mathrm{Sq}^2 v) = \mathrm{Sq}^2(\delta_2 v) = \mathrm{Sq}^2(\beta) = 0$, and $v'^2 = 0$ iff $\mathrm{Sq}^2(\beta') = 0$. Let $w_2 \in H^2(RP^2; Z_2)$ and $w_2' \in H^2(RP^2; Z_2)$ be the second Stiefel-Whitney classes of the bundles $\varepsilon \oplus \varepsilon \oplus \eta$ and $\eta \oplus \eta \oplus \eta$. Then $w_2 = 0$ and $w_2' \neq 0$. (The Stiefel-Whitney classes of $\varepsilon \oplus \varepsilon \oplus \eta$ and $\eta \oplus \eta \oplus \eta$ are $(1 + \alpha)$ and $(1 + \alpha)^3 = 1 + \alpha + \alpha^2$ respectively.) $\mathrm{Sq}^2(\beta) = \beta \cup w_2 = 0$ and $\mathrm{Sq}^2(\beta') = \beta' \cup w_2' \neq 0$. (See [7, §VII].) Hence $\beta^2 = 0$ and $\beta'^2 \neq 0$. In summary we have the following propositions.

PROPOSITION V.17. *The Z_2-cohomology ring of L_2 is generated by two elements $u \in H^1(L_2; Z_2)$ and $v \in H^2(L_2; Z_2)$ such that $u^3 = 0$ and $v^2 = 0$.*

PROPOSITION V.18. *The Z_2-cohomology ring of L_2' is generated by two elements $u' \in H^1(L_2'; Z_2)$ and $v' \in H^2(L_2'; Z_2)$ such that $u'^3 = 0$, $v'^2 = u'^2v'$ and $v'^3 = 0$.*

In particular, the mod 2 intersection matrices of L_2 and L_2' are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

respectively. They are clearly not equivalent. Hence L_2 and L_2' are not homotopically equivalent.

It follows from *Corollary* II.3.14 that $L_2 \# (k-1)(S^2 \times S^2)$ and $L_2' \# (k-1)(S^2 \times S^2)$ are the covering spaces of L_{2k} and L_{2k}' respectively. Let $\pi: L_2 \# (k-1)(S^2 \times S^2) \to L_{2k}$ and $\pi': L_2' \# (k-1)(S^2 \times S^2) \to L_{2k}'$ be the covering map. Let $\mu_i: H^i(L_2 \# (k-1)(S^2 \times S^2); Z) \to H^i(L_{2k}; Z)$ and $\mu_i': H^i(L_2' \# (k-1)(S^2 \times S^2); Z) \to H^i(L_{2k}'; Z)$, $i = 0, 1, 2, 3, 4$, be the transfer homomorphisms. Then

$$\mu_i \pi^* = \cdot k \colon H^i(L_{2k}; Z) \to H^i(L_{2k}; Z),$$

and

$$\mu_i' \pi'^* = \cdot k \colon H^i(L_{2k}'; Z) \to H^i(L_{2k}'; Z).$$

(See [1, Chapter III, §7].) By an elementary group theory argument we have the following:

(1) The homomorphisms

$$\begin{array}{ccc} \pi_2^*\colon H^2(L_{2k}; Z) & \longrightarrow & H^2(L_2 \#(k-1)(S^2 \times S^2); Z) \\ \Big\| & & \Big\| \\ Z_{2k} & \longrightarrow & H^2(L_2; Z) \oplus H^2(\#(k-1)(S^2 \times S^2); Z) = Z_2 \oplus (2k-2)Z, \end{array}$$

and

$$\begin{array}{ccc} \pi_2'^*\colon H^2(L_{2k}'; Z) & \longrightarrow & H^2(L_2' \#(k-1)(S^2 \times S^2); Z) \\ \Big\| & & \Big\| \\ Z_{2k} & \longrightarrow & H^2(L_2'; Z) \oplus H^2(\#(k-1)(S^2 \times S^2); Z) = Z_2 \oplus (2k-2)Z, \end{array}$$

map $H^2(L_{2k}; Z)$ and $H^2(L_{2k}'; Z)$ onto the submodules $H^2(L_2; Z)$ and $H^2(L_2'; Z)$ of

$$H^2(L_2 \# (k-1)(S^2 \times S^2); Z) \quad \text{and} \quad H^2(L_2' \# (k-1)(S^2 \times S^2); Z)$$

respectively.

(2) The homomorphisms

$$\begin{array}{ccc} \pi_3^*\colon H^3(L_{2k}; Z) & \longrightarrow & H^3(L_2 \#(k-1)(S^2 \times S^2); Z) \\ \Big\| & & \Big\| \\ Z_{2k} & \longrightarrow & H^3(L_2; Z) = Z_2, \end{array}$$

and

$$\begin{array}{ccc} \pi_3'^*\colon H^3(L'_{2k}; Z) & \longrightarrow & H^3(L'_2 \#(k-1)(S^2 \times S^2); Z) \\ \Big\| & & \Big\| \\ Z_{2k} & \longrightarrow & H^3(L'_2; Z) = Z_2, \end{array}$$

are surjective.

Passing to Z_2-cohomology by the Universal Coefficient Theorem, the following assertions are immediate:

(1) $\pi_1^*\colon H^1(L_{2k}; Z_2) \to H^1(L_2 \# (k-1)(S^2 \times S^2); Z_2)$ and

$$\pi_3^*\colon H^3(L_{2k}; Z_2) \to H^3(L_2 \# (k-1)(S^2 \times S^2); Z_2)$$

are isomorphisms.

(2) $\pi_1'^*\colon H^1(L'_{2k}; Z_2) \to H^1(L'_2 \# (k-1)(S^2 \times S^2); Z_2)$ and

$$\pi_3'^*\colon H^3(L'_{2k}; Z_2) \to H^3(L'_2 \# (k-1)(S^2 \times S^2); Z_2)$$

are isomorphisms.

(3) Consider the exact sequences

$$\begin{array}{ccccccccc} 0 \to & H^2(L_{2k}; Z) \otimes Z_2 & \longrightarrow & H^2(L_{2k}; Z_2) & \longrightarrow & H^3(L_{2k}; Z) * Z_2 & \to 0 & & \\ & \Big\downarrow \pi_2^* \otimes \mathrm{id} & & \Big\downarrow \pi_2^* & & \searrow \pi_2^* * \mathrm{id} & & & \\ 0 \to & H^2(L_2 \# (k-1)(S^2 \times S^2); Z) \otimes Z_2 & \longrightarrow & H^2(L_2 \# (k-1)(S^2 \times S^2); Z_2) & \longrightarrow & H^2(L_2 \# (k-1)(S^2 \times S^2); Z) * Z_2 & \to 0 & & \\ & \Big\downarrow = & & & & \Big\downarrow = & & & \\ & (H^2(L_2; Z) \otimes Z_2) \oplus H^2(\#(k-1)(S^2 \times S^2); Z) \otimes Z_2 & & & & (H^3(L_2; Z) * Z_2) \oplus H^3(\#(k-1)(S^2 \times S^2); Z) * Z_2 & & & \end{array}$$

Note that the first and third vertical maps are isomorphisms onto the submodules $H^2(L_2; Z_2) \otimes Z_2$ and $H^3(L_2; Z) * Z_2$. By the commutativity of the diagram we assert that

$$\pi_2^*\colon H^2(L_{2k}; Z_2) \to H^2(L_2 \# (k-1)(S^2 \times S^2); Z_2)$$

is an isomorphism which maps $H^2(L_{2k}; Z_2)$ onto the submodule $H^2(L_2; Z_2)$ of $H^2(L_2 \# (k-1)(S^2 \times S^2); Z_2)$.

(4) Similarly, $\pi_2'^*: H^2(L'_{2k}; Z_2) \to H^2(L'_2 \# (k-1)(S^2 \times S^2); Z_2)$ is also an isomorphism which maps $H^2(L'_{2k}; Z_2)$ onto the submodule $H^2(L'_2; Z_2)$ of $H^2(L'_2 \# (k-1)(S^2 \times S^2); Z_2)$.

In conclusion we have:

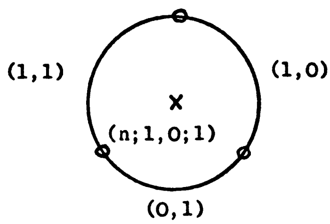
THEOREM V.19.

$$H^*(L_{2k}; Z_2) = H^*(L_2; Z_2) \quad \textit{and} \quad H^*(L'_{2k}; Z_2) = H^*(L'_2; Z_2)$$

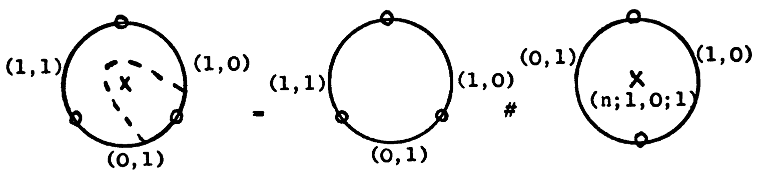
for any positive integer k.

As in *Proposition* V.18, we see that L_{2k} and L'_{2k} have nonequivalent mod 2 quadratic forms. Therefore L_{2k} and L'_{2k} are not homotopically equivalent. This finishes the proof of Theorem V.2.

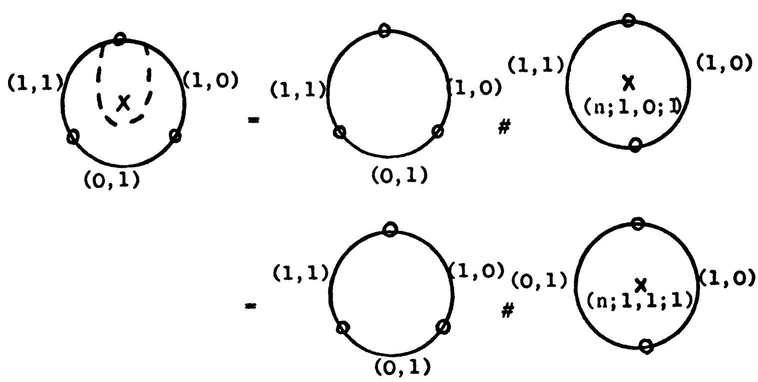
Let M be the T^2-manifold corresponding to the orbit space



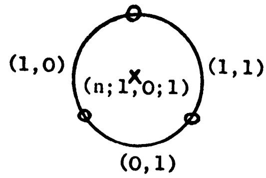
Then M can be decomposed in two ways:



and



That is $M = -CP^2 \# L_n = -CP^2 \# L'_n$. Similarly, by considering the T^2-manifold with orbit space



we have $CP^2 \# L_n = CP^2 \# L'_n$.

THEOREM V.20. *$CP^2 \# L_n = CP^2 \# L'_n$ and $-CP^2 \# L_n = -CP^2 \# L'_n$ for all integers $n \geqslant 2$.*

VI. The connected sum decomposition theorem. Combining *Theorem* III.3, *Theorem* IV.3, *Theorem* V.1, *Theorem* V.2, I.6 and the fact that the manifolds S^4, $S^2 \times S^2$, $S^1 \times S^3$, L_n and L'_n ($n = 2, 3, \ldots$) admit orientation reversing self-homeomorphisms, we have the following:

THEOREM VI.1. *Every T^2-manifold with fixed points can be decomposed into a connected sum of copies of S^4, $S^2 \times S^2$, CP^2, $-CP^2$, $S^1 \times S^3$, L_n and L'_n ($n = 2, 3, \ldots$).*

Unfortunately this connected sum decomposition is not unique. For example:

(1) $CP^2 \# S^2 \times S^2 = CP^2 \# -CP^2 \# CP^2$,

(2) $CP^2 \# L_n = CP^2 \# L'_n$ for any integer $n \geqslant 2$,

(3) $-CP^2 \# L_n = -CP^2 \# L'_n$ for any integer $n \geqslant 2$, and

(4) $L_n = L'_n$ for $n = 3, 5, 7, \ldots$.

In this section we shall give a normal form of this connected sum decomposition, which is unique topologically. In fact, the above homeomorphic relations (1) to (4) are the entire "basic" homeomorphic relations in this decomposition. The topological classification of T^2-manifolds with fixed points is therefore complete.

DEFINITION. Let M be a T^2-manifold with fixed points. A connected sum decomposition of M in terms of S^4, $S^2 \times S^2$, $S^1 \times S^3$, CP^2, $-CP^2$, L_n and L'_n ($n = 2, 3, \ldots$) is called a normal decomposition, if the number of copies of S^4, CP^2, $-CP^2$ and L'_n ($n = 2, 3, \ldots$), (which will be denoted by $N(S^4)$, $N(CP^2)$, $N(-CP^2)$, $N(L'_n)$), satisfy the following conditions:

(a) $N(S^4) = 1$,

(b) $N(-CP^2) = 0$ or $N(CP^2) = N(-CP^2) = 1$,

(c) if $N(CP^2) \neq 0$ then $N(L'_n) = 0$ for all $n = 2, 3, \ldots$,

(d) $N(L'_n) = 0$ for all $n = 3, 5, 7, \ldots$.

THEOREM II.4.2. *Every T^2-manifold with fixed points has a unique normal decomposition.*

PROOF. Let M be a T^2-manifold with fixed points. By *Theorem* VI.1 we may assume $M = S^4 \# a(S^2 \times S^2) \# b(CP^2) \# d(-CP^2) \# e(S^1 \times S^3) \# J_2(L_2) \# K_2(L'_2) \# J_3(L_3) \# K_3(L'_3) \# \cdots \# J_t(L_t) \# K_t(L'_t)$, and $b \geq d$. (This is possible because M is orientable,we can choose a suitable orientation on M so $b \geq d$.) By the homeomorphic relations (1) to (4) above, we can easily change the above decomposition into a normal form.

Suppose

$$M = S^4 \# a(S^2 \times S^2) \# b(CP^2) \# d(-CP^2) \# e(S^1 \times S^3) \\ \# J_2(L_2) \# K_2(L'_2) \# J_3(L_3) \# J_4(L_4) \# K_4(L'_4) \# \cdots \# J_t(L_t) \# K_t(L'_t)$$

is a normal decomposition. We shall show that this normal form is unique, or equivalently that the numbers $a, b, d, e, J_2, K_2, J_3, J_4, K_4, \ldots, J_t, K_t$ are topological invariants. The proof is divided into three steps.

(1) The invariance of a, b and d. Note that $\text{Index}(M) = b - d \geq 0$. Suppose $b - d = 0$. It follows from the definition of normal decomposition that either $b = d = 0$ or $b = d = 1$. Observe that M is of type II if $b = d = 0$, and it is of type I if $b = d = 1$. Suppose $b - d > 0$, then $d = 0$ and $\text{Index}(M) = b$. By the invariance of the type and index of M, the number b and d are invariants. Moreover,

$$rkH^2(M; Z) = 2a + b + d.$$

Since $rkH^2(M; Z)$, b and d are all invariants, a is also an invariant.

(2) The invariance of $e, J_2 + K_2, J_3, J_4 + K_4, \ldots, J_t + K_t$. Observe that

$$\pi_1(M) = \underbrace{Z * \cdots * Z}_{e} * \underbrace{Z_2 * \cdots * Z_2}_{J_2+K_2} * \underbrace{Z_3 * \cdots * Z_3}_{J_3} * \cdots * \underbrace{Z_t * \cdots * Z_t}_{J_t+K_t},$$

(where "$*$" stands for the free product), and the above free product decomposition of $\pi_1(M)$ is indecomposable. It follows from the uniqueness of the indecomposable free product of decomposition of $\pi_1(M)$ that the numbers $e, J_2 + K_2, J_3, \ldots, J_t + K_t$ are invariants. (For the uniqueness of the free product decomposition, see [4, §35].)

(3) The invariance of $J_2, K_2, J_4, K_4, \ldots, J_t, K_t$.

Case I:$b \neq 0$. It follows from the definition of normal decomposition that the integers $K_2, K_4, \ldots, K_t$ are all zero. So $J_2 = J_2 + K_2, J_4 = J_4 + K_4, \ldots, J_t = J_t + K_t$. By (2) the theorem is proved.

Case II: $b = 0$. Let $\hat{M} = S^4 \# a'(S^2 \times S^2) \# e'(S^1 \times S^3) \# J'_2(L_2) \# K'_2(L'_2) \# J'_3(L_3) \# \cdots \# J'_t(L_t) \# K'_t(L'_t)$ be a normal decomposition of a manifold $\hat{M}$, and f: $M \to \hat{M}$ be a homeomorphism. It follows from (1) and (2) above that $a' = a, e' = e, J'_3 = J_3, J'_5 = J_5, \ldots,$ and $J'_2 + K'_2 = J_2$

$+ K_2, \ldots, J_t' + K_t' = J_t + K_t$. We need to show that $J_2' = J_2, J_4' = J_4, \ldots, J_t' = J_t$.

For an even number s, $2 \leqslant s \leqslant t$, and an integer i, $1 \leqslant i \leqslant J_s$ $(1 \leqslant i \leqslant J_s')$, let $L_{s,i}$ denote the ith copy of L_s in the normal decomposition of $M(\hat{M})$. Similarly, for an even number s, $2 \leqslant s \leqslant t$, and an integer i, $1 \leqslant i \leqslant K_s$ $(1 \leqslant i \leqslant K_s')$. Let $L_{s,i}'$ denote the ith copy of L_s' in the normal decomposition of $M(\hat{M})$. Define $u_{s,i} \in H^1(L_{s,i}; Z_2)$ and $v_{s,i} \in H^2(L_{s,i}; Z_2)$ such that $u_{s,i}$ generates $H^1(L_{s,i}; Z_2)$, and $(u_{s,i})^2$ and $v_{s,i}$ generate $H^2(L_{s,i}; Z_2)$. Similarly, define $u_{s,i}' \in H^1(L_{s,i}'; Z_2)$ and $v_{s,i}' \in H^2(L_{s,i}'; Z_2)$. Let f_A^i: $H^i(\hat{M}; A) \to H^i(M; A)$, $i = 0, 1, 2, 3, 4$, be the isomorphisms induced by the homeomorphism f. For

$$
\begin{aligned}
f_{Z_2}^1&: H^1(\hat{M}; Z_2)\\
&= e'(H^1(S^1 \times S^3; Z_2)) \oplus J_2'(H^1(L_2; Z_2)) \oplus K_2'(H^1(L_2'; Z_2))\\
&\qquad \oplus \cdots \oplus J_t'(H^1(L_t; Z_2)) \oplus K_t'(H^1(L_t'; Z_2)) \to H^1(M; Z_2)\\
&= e(H^1(S^1 \times S^3; Z_2)) \oplus J_2(H^1(L_2; Z_2)) \oplus K_2(H^1(L_2'; Z_2))\\
&\qquad \oplus \cdots \oplus J_t(H^1(L_t; Z_2)) \oplus K_t(H^1(L_t'; Z_2)),
\end{aligned}
$$

we have the following:

SUBLEMMA. Given any even number s, $2 \leqslant s \leqslant t$, and an integer i, $1 \leqslant i \leqslant J_s'$, then $f_{Z_2}^1(u_{s,i}) = u_{s,j}$ for some integer j, $1 \leqslant j \leqslant J_s$. Similarly, given any even number s, $2 \leqslant s \leqslant t$, and an integer i, $1 \leqslant i \leqslant K_s'$, then $f_{Z_2}^1(u_{s,i}') = u_{s,j}'$ for some integer j, $1 \leqslant j \leqslant K_s$.

Assuming this sublemma, it follows immediately that $J_2' = J_2, J_4' = J_4, \ldots, J_t' = J_t$ and $K_2' = K_2, K_4' = K_4, \ldots, K_t' = K_t$. The theorem is proved.

PROOF OF THE SUBLEMMA. Let $\alpha_{s,i}$ and $\alpha_{s,i}'$ be generators of $\pi_1(L_{s,i})$ and $\pi_1(L_{s,i}')$ respectively. Let

$$
\begin{aligned}
f_\#: \pi_1(M) &= \underbrace{\pi_1(S^1 \times S^3) * \cdots * \pi_1(S^1 \times S^3)}_{e} * \pi_1(L_{2,1}) * \cdots * \pi_1(L_{2,J_2})\\
&\qquad * \pi_1(L_{2,1}') * \cdots * \pi_1(L_{2,K_2}') * \cdots * \pi_1(L_{t,1}) * \cdots * \pi_1(L_{t,J_t})\\
&\qquad * \pi_1(L_{t,1}') * \cdots * \pi_1(L_{t,K_t}') \to \pi_1(\hat{M})\\
&= \underbrace{\pi_1(S^1 \times S^3) * \cdots * \pi_1(S^1 \times S^3)}_{e} * \pi_1(L_{2,1}) * \cdots * \pi_1(L_{2,J_2'})\\
&\qquad * \pi_1(L_{2,1}') * \cdots * \pi_1(L_{2,K_2'}') * \cdots * \pi_1(L_{t,1}) * \cdots\\
&\qquad * \pi_1(L_{t,J_t'}) * \pi_1(L_{t,1}') * \cdots * \pi_1(L_{t,K_t'}')
\end{aligned}
$$

be the isomorphism induced by the homeomorphism f. Since $f_\#$ is an isomorphism, $f_\#$ preserves the order and divisibility of every element. It follows that for integers s, j, $2 \leqslant s \leqslant t$, $1 \leqslant j \leqslant J_s$, $f_\#(\alpha_{s,j})$ is a conjugate of some element $\alpha_{s,i}$ or $\alpha'_{s,i}$ in $\pi_1(\hat{M})$, where i is an integer such that $1 \leqslant i \leqslant J_s$ or $1 \leqslant i \leqslant K_s$. Passing to $f_1: H_1(M;Z) \to H_1(\hat{M};Z)$ by Hurewicz Homomorphism Theorem, then to $f_1: H_1(M;Z_2) \to H_1(\hat{M};Z_2)$ and $f^1_{Z_2}: H^1(\hat{M},Z_2) \to H^1(M;Z_2)$ by the Universal Coefficient Theorem, we have $f^1_{Z_2}(u_{s,i})$ is either $u_{s,j}$ or $u'_{s,j}$, for some integer j such that $1 \leqslant j \leqslant J_s$ or $1 \leqslant j \leqslant K_2$. Similarly, we have $f^1_{Z_2}(u'_{s,i})$ is either $u_{s,j}$ or $u'_{s,j}$ for some integers j such that $1 \leqslant j \leqslant J_s$ or $1 \leqslant j \leqslant K_s$.

Suppose $f^1_{Z_2}(u_{s,i}) = u'_{s,j}$. Then $f^2_{Z_2}(u^2_{s,i}) = u'^2_{s,j}$. Consider $f^2_Z: H^2(\hat{M};Z) \to H^2(M;Z)$, since f^2_Z must map the torsion part of $H^2(\hat{M};Z)$ isomorphically onto the torsion part of $H^2(M;Z)$, it follows that f^2_Z maps the submodule $J'_2(H^2(L_2;Z)) \oplus K'_2(H^2(L'_2;Z)) \oplus \cdots \oplus J'_t(H^2(L_t;Z)) \oplus K'_t(H^2(L'_t;Z))$ of $H^2(\hat{M};Z)$ onto the submodule $J_2(H^2(L_2;Z)) \oplus K_2(H^2(L'_2;Z)) \oplus \cdots \oplus J_t(H^2(L_t;Z)) \oplus K_t(H^2(L'_t;Z))$ of $H^2(M;Z)$. Passing to mod 2 cohomology by the Universal Coefficient Theorem, we have that $f^2_{Z_2}$ maps the submodule

$$J'_2(H^2(L_2;Z_2)) \oplus K'_2(H^2(L'_2;Z_2)) \oplus \cdots \oplus J'_t(H^2(L_t;Z_2)) \oplus K'_t(H^2(L'_t;Z_2))$$

of $H^2(\hat{M};Z_2)$ onto the submodule

$$J_2(H^2(L_2;Z_2)) \oplus K_2(H^2(L'_2;Z_2)) \oplus \cdots \oplus J_t(H^2(L_t;Z_2)) \oplus K_t(H^2(L'_t;Z_2))$$

of $H^2(M;Z_2)$. Let

$$f^2_{Z_2}(v_{s,i}) = \sum_{d=2}^{t}\left(\sum_{i=1}^{J_d} \gamma_{d,u} u^2_{d,i}\right) + \sum_{d=2}^{t}\left(\sum_{i=1}^{J_d} \delta_{d,i} v_{d,i}\right) + \sum_{d=2}^{t}\left(\sum_{i=1}^{K_d} \gamma'_{d,i} u'^2_{d,i}\right) + \sum_{d=2}^{t}\left(\sum_{i=1}^{K_d} \delta'_{d,i} v'_{d,i}\right),$$

where $\gamma_{d,i}, \delta_{d,i}, \gamma'_{d,i}, \delta'_{d,i}$ are either 0 or 1. It is easily seen that in $H^*(M;Z_2)$, $u^2_{d,i} \cup f^2_{Z_2}(v_{s,i}) = \delta_{d,i}$, and $u'^2_{d,i} \cup f^2_{Z_2}(v_{s,i}) = \delta'_{d,i}$. Since $f^4(u^2_{s,i} \cup v_{s,i}) = f^2_{Z_2}(u^2_{s,i}) \cup f^2_{Z_2}(v_{s,i}) = u'^2_{s,j} \cup f^2_{Z_2}(v_{s,i}) = \delta'_{s,j}$, and $u^2_{s,i} \cup v_{s,i} = 1$, it follows that $\delta'_{s,j} = 1$. Moreover, since $u^2_{d,b} \cup v_{s,i} = 0$ if $d \neq s$ or $b \neq i$, and $u'^2_{d,b} \cup v_{s,i} = 0$ for all d and b, a similar argument shows that $\delta_{d,b} = 0$ for all d and b, and $\delta'_{d,b} = 0$ if $d \neq s$ or $b \neq j$. In conclusion: if $f^1_{Z_2}(u_{s,i}) = u'_{s,j}$, then

$$f_{Z_2}^2(v_{s,i}) = v'_{s,j} + \sum_{d=2}^{t}\left(\sum_{i=1}^{J_d} \gamma_{d,i} u_{d,i}^2\right) + \sum_{d=2}^{t}\left(\sum_{i=1}^{K_d} \gamma'_{d,i} u'^2_{d,i}\right).$$

Observe that $f_{Z_2}^2(v_{s,i}) \cup f_{Z_2}^2(v_{s,i}) = v'_{s,j} \cup v'_{s,j} = 1$. But $f_{Z_2}^2(v_{s,i}) \cup f_{Z_2}^2(v_{s,i}) = f_{Z_2}^4(v_{s,i} \cup v_{s,i}) = f_{Z_2}^4(0) = 0$. This is a contradiction, hence $f_{Z_2}^1(u_{s,i})$ must equal $u_{s,j}$ for some integer j such that $1 \leqslant j \leqslant J_2$. Using the same argument we can show that $f_{Z_2}^1(u'_{s,i}) = u'_{s,j}$ for some integer j such that $1 \leqslant j \leqslant K_s$. The assertion is proved.

References

1. G. Bredon, *Introduction to compact transformations groups*, Academic Press, New York, 1972.

2. P. Conner and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, 1969), Markham, Chicago, 1970, pp. 227–264. MR **42** #6839.

3. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment Math. Helv. **36** (1961), 47–82. MR **26** #3069.

4. A. G. Kuroš, *Theory of groups*, Vol. 2, 2nd ed., GITTL, Moscow, 1953; Chelsea, New York, 1956. MR **15**, 501; **18**, 188.

5. J. Milnor, *On simply connected 4-manifolds*, Internat. Sympos. Algebraic Topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 122–128. MR **21** #2240.

6. ———, *A procedure for killing homotopy groups of differentiable manifolds*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R.I., 1961, pp. 39–55. MR **24** #A556.

7. J. Milnor and J. Stasheff, *Characteristic classes*, Princeton Univ. Press, Princeton, N.J., 1974.

8. P. Orlik and F. Raymond, *Actions of the torus on 4-manifolds*. I, Trans. Amer. Math. Soc. **152** (1970), 531–559. MR **42** #3808.

9. ———, *Actions of the torus on 4-manifolds*. II, Topology **13** (1974), 89–112. MR **50** #1274.

10. F. Raymond, *Classification of the actions of the circle on 3-manifolds*, Trans. Amer. Math. Soc. **131** (1968), 51–78. MR **36** #2169.

11. H. Zieschang, *On torus fiberings over surfaces*, Mat. Zametki **5** (1969), 569–576 = Math. Notes **5** (1969), 341–345. MR **40** #2085.

Department of Mathematics, University of Georgia, Athens, Georgia 30602